Kontsevich's graph complex, GRT, and the deformation complex of the sheaf of polyvector fields

V.A. Dolgushev, C.L. Rogers, and T.H. Willwacher

To the memory of Boris Vasilievich Fedosov

Abstract

We generalize Kontsevich's construction [40] of L_∞-derivations of polyvector fields from the affine space to an arbitrary smooth algebraic variety. More precisely, we construct a map (in the homotopy category) from Kontsevich's graph complex to the deformation complex of the sheaf of polyvector fields on a smooth algebraic variety. We show that the action of the Deligne-Drinfeld elements of the Grothendieck-Teichmüller Lie algebra on the cohomology of the sheaf of polyvector fields coincides with the action of odd components of the Chern character. Using this result, we deduce that the $\hat{A}$-genus in the Calaque-Van den Bergh formula [13] for the isomorphism between harmonic and Hochschild structures can be replaced by a generalized $\hat{A}$-genus.

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1 Introduction

Inspired by Grothendieck's lego-game from [32], V. Drinfeld introduced in [25] a pro-unipotent algebraic group which he called the Grothendieck-Teichmüller group **GRT**. This group is closely connected with the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, it appears naturally in the study of moduli of algebraic curves, solutions of the Kashiwara-Vergne problem [1], theory of motives [7], [16], [27] and formal quantization procedures [19], [20], [46], [47]. The Lie algebra $\mathfrak{grt}$ of **GRT** carries a natural grading by positive integers. Furthermore, according

to [25, Proposition 6.3], $\mathfrak{grt}$ has a non-zero vector σ_n for every odd degree $n \geq 3$. We call σ_n's the Deligne-Drinfeld elements of $\mathfrak{grt}$.

In paper [47], the third author established a link between the graph complex GC introduced in [40] by M. Kontsevich and the Lie algebra $\mathfrak{grt}$ of the Grothendieck-Teichmüller group. More precisely, in [47], it was shown that

$$H^0(\mathsf{GC}) \cong \mathfrak{grt}\,. \tag{1.1}$$

In paper [41], M. Kontsevich conjectured that the Grothendieck-Teichmüller group reveals itself in the extended moduli [4] of deformations of an algebraic variety X via the action of odd components of the Chern character of X on the cohomology of the sheaf of polyvector fields

$$H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}})\,. \tag{1.2}$$

In this paper, we use the isomorphism (1.1) to establish this fact for an arbitrary smooth algebraic variety X over an algebraically closed field $\mathbb{K}$ of characteristic zero.

More precisely, we define a map (in the homotopy category of dg Lie algebras) from GC to the deformation complex of the sheaf of polyvector fields $\mathcal{T}_{\mathrm{poly}}$ on an arbitrary smooth algebraic variety X. This result generalizes Kontsevich's construction [40, Section 5] from the case of affine space to the case of an arbitrary smooth algebraic variety.

Using a link [47] between the graph complex GC and the deformation complex of the operad Ger, we prove that every cocycle $\gamma \in \mathsf{GC}$ gives us a derivation of the Gerstenhaber algebra (1.2).

Combining these results with the isomorphism (1.1), we get a natural action of the Lie algebra $\mathfrak{grt}$ on the cohomology (1.2) of the sheaf of polyvector fields. In addition, we deduce that this action is compatible with the Gerstenhaber algebra structure on (1.2).

We show that the action of the Deligne-Drinfeld element σ_n (n odd ≥ 3) of $\mathfrak{grt}$ on (1.2) is given by a non-zero multiple of the contraction with the n-th component of the Chern character of X. This result confirms that the Grothendieck-Teichmüller group indeed reveals itself in the extended moduli of deformations of X in the way predicted by M. Kontsevich in [41]. Our results imply that the contraction of polyvector fields with any odd component of the Chern character induces a derivation of (1.2) with respect to the cup-product. This statement was formulated in [41, Theorem 9] without a proof.

We prove that the $\hat{A}$-genus in the Calaque-Van den Bergh formula [13] for the isomorphism between harmonic and Hochschild structures can be replaced by a generalized $\hat{A}$-genus.

We give examples of algebraic varieties for which odd components of the Chern character act non-trivially on (1.2). In particular, using Theorem 8.1, we showed that smooth Calabi-Yau complete intersections in projective spaces provide us with a large supply of non-trivial representations of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$. This situation is strikingly different from what we have in the classical Duflo theory and in the classical Poisson geometry. Indeed, as remarked by M. Duflo (see [41, Section 4.6]), the action of $\mathfrak{grt}$ on Duflo isomorphisms is trivial for all (non-graded) Lie algebras. Furthermore, the authors are still unaware of any instance of a (non-graded) Poisson structure on which $\mathfrak{grt}$ acts non-trivially.

Finally, we show how Corollary 8.2 allows us to get some information about the Gerstenhaber algebra structure on (1.2) when X is a complete intersection in a projective space.

Recent related results We would like to mention two papers [2] and [38] in which similar results were obtained.

In paper [2], J. Alm and S. Merkulov proved that, for an arbitrary formal Poisson structure π on a smooth real manifold M and an arbitrary element g of the group **GRT**, the Poisson cohomology $H^\bullet(M, g(\pi))$ of $g(\pi)$ is isomorphic to the Poisson cohomology $H^\bullet(M, \pi)$ of π as a graded associative algebra.

In paper [38], C. Jost described a large class of L_∞-automorphisms of the Schouten algebra of polyvector fields on $\mathbb{R}^d$ which can be "extended" to L_∞-automorphisms of the Schouten algebra of polyvector fields on an arbitrary smooth real manifold. Combining this result with the isomorphism (1.1), C. Jost constructed an action the group **GRT** by L_∞-automorphisms on the Schouten algebra of polyvector fields on an arbitrary smooth real manifold.

Structure of the paper In the remaining subsections of the Introduction, we fix notation and conventions.

Sections 2 and 3 are devoted to the Fedosov resolution of the sheaf of tensor fields on a smooth algebraic variety.

The key idea of this construction [17], [18] has various incarnations and it is often referred to as the Gelfand-Fuchs trick [28] or Gelfand-Kazhdan formal geometry [29] or mixed resolutions [48].

The version given here is a modification of the construction proposed in [45] by M. Van den Bergh. The important advantage of our version is that we managed to streamline Van den Bergh's approach by avoiding completely the use of formal schemes and the use of jets. We believe that our modification of Van den Bergh's construction will be useful far beyond the scope of our paper.

In Section 4, we describe a convenient explicit representative of the Atiyah class of X in the Fedosov resolution of the tensor algebra. In this section, we also observe that the Fedosov resolution allows us to represent this class by a global section of some sheaf unlike the conventional representative which is given by a 1-cochain in the Čech complex.

In Section 5, we recall the operad Gra, the full graph complex fGC, and Kontsevich's graph complex [40, Section 5] GC. We state the results of the third author from [47] which are used later in the text and introduce a couple of dg Lie algebras related to the full graph complex fGC.

Section 6 is devoted to the construction of a map Θ of dg Lie algebras from Kontsevich's graph complex GC to the deformation complex of the dg sheaf $\mathcal{FR}$ which is quasi-isomorphic to the sheaf of polyvector fields on X. In this section, we consider the sheaf $\mathcal{FR}$ primarily with the Schouten-Nijenhuis bracket forgetting the cup product structure. However, in technical Section 6.1, we extend the map Θ to a map from an auxiliary dg Lie algebra linked to fGC to the deformation complex of $\mathcal{FR}$, where $\mathcal{FR}$ is viewed as a sheaf of dg Gerstenhaber algebras.

In Section 7, we prove that for every cocycle $\gamma \in \mathsf{GC}$ the cocycle $\Theta(\gamma)$ induces a derivation of the Gerstenhaber algebra $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$.

In Section 8, we give a geometric description of the action of the Deligne-Drinfeld elements of $\mathfrak{grt}$ on the Gerstenhaber algebra $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$. In this section, we also prove that

the contraction with odd components of the Chern character induces derivations of the Gerstenhaber algebra $H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}})$.

In Section 9, we generalize the result [13] of D. Calaque and M. Van den Bergh on harmonic and Hochschild structures of a smooth algebraic variety.

In Section 10, we give several examples which show that Theorem 8.1 and Theorem 9.2 are non-trivial. Many of these examples can be found among complete intersections in a projective space.

At the end of the paper we have several appendices.

In Appendix A, we briefly recall the notion of a homotopy O-algebra and the notion of the deformation complex of an O-algebra.

In Appendix B, we recall necessary constructions related to sheaves of algebras over an operad. More precisely, we review the Thom-Sullivan normalization and use it to define derived global sections for a dg sheaf $\mathcal{A}$ of operadic algebras and the deformation complex of $\mathcal{A}$. Although the Thom-Sullivan normalization is extremely convenient for proving general facts about derived global sections and the deformation complex, in the bulk of our paper, we use the conventional Cech resolution. This use is justified by Propositions B.5 and B.8.

In Appendix C, we briefly recall twisting of shifted Lie algebras and Gerstenhaber algebras by Maurer-Cartan elements. In this appendix, we also extend the twisting operation to a subspace of cochains in the deformation complexes of such algebras.

Most of the material given in the appendices is standard and well known to specialists. However, various statements are hard to find in the literature in the desired generality. So we added these appendices for convenience of the reader.

1.1 Notation and conventions

Throughout the paper $\mathbb{K}$ is an algebraically closed field of characteristic zero.

The notation S_n is reserved for the symmetric group on n letters and $\mathrm{Sh}_{p_1,\dots,p_k}$ denotes the subset of $(p_1,\dots,p_k)$-shuffles in S_n, i.e. $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),$$
$$\sigma(p_1+1) < \sigma(p_1+2) < \cdots < \sigma(p_1+p_2),$$
$$\dots$$
$$\sigma(n-p_k+1) < \sigma(n-p_k+2) < \cdots < \sigma(n)\,.$$

For algebraic structures considered in this paper, we use the following symmetric monoidal categories:

- the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces,
- the category of (possibly) unbounded cochain complexes of $\mathbb{K}$-vector spaces,
- the category of sheaves of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces,
- the category of sheaves of (possibly) unbounded cochain complexes of $\mathbb{K}$-vector spaces.

In particular, we frequently use the ubiquitous combination "dg" (differential graded) to refer to algebraic objects in the category of cochain complexes or the category of sheaves of cochain complexes. We often consider a graded vector space (resp. a sheaf of graded vector spaces) as the cochain complex (resp. the sheaf of cochain complexes) with the zero differential.

For a homogeneous vector v in a cochain complex $\mathcal{V}$, $|v|$ denotes the degree of v. Furthermore, we denote by $\mathbf{s}$ (resp. $\mathbf{s}^{-1}$) the operation of suspension (resp. desuspension), i.e.

$$(\mathbf{s}\,\mathcal{V})^{\bullet} = \mathcal{V}^{\bullet-1}\,, \qquad (\mathbf{s}^{-1}\,\mathcal{V})^{\bullet} = \mathcal{V}^{\bullet+1}\,.$$

We reserve the notation $S(\mathcal{V})$ (resp. $\underline{S}(\mathcal{V})$) for the symmetric algebra (resp. the truncated symmetric algebra) in $\mathcal{V}$:

$$S(\mathcal{V}) = \mathbb{K} \,\oplus\, \bigoplus_{n\ge 1} \big(\mathcal{V}^{\otimes\, n}\big)_{S_n}\,, \tag{1.3}$$

$$\underline{S}(\mathcal{V}) = \bigoplus_{n\ge 1} \big(\mathcal{V}^{\otimes\, n}\big)_{S_n}\,. \tag{1.4}$$

The notation Lie (resp. Com, Ger) is reserved for the operad governing Lie algebras (resp. commutative (and associative) algebras without unit, Gerstenhaber algebras without unit). Dually, the notation coLie (resp. coCom) is reserved for the cooperad governing Lie coalgebras (resp. cocommutative (and coassociative) coalgebras without counit).

For an operad O (resp. a cooperad C) and a cochain complex (or a sheaf of cochain complexes) $\mathcal{V}$, the notation $O(\mathcal{V})$ (resp. $C(\mathcal{V})$) is reserved for the free O-algebra (resp. cofree C-coalgebra). Namely,

$$O(\mathcal{V}) := \bigoplus_{n\ge 0} \Big(O(n)\otimes \mathcal{V}^{\otimes\, n}\Big)_{S_n}\,, \tag{1.5}$$

$$C(\mathcal{V}) := \bigoplus_{n\ge 0} \Big(C(n)\otimes \mathcal{V}^{\otimes\, n}\Big)^{S_n}\,. \tag{1.6}$$

For example[1]

$$\mathsf{coCom}(\mathcal{V}) = \underline{S}(\mathcal{V})\,. \tag{1.7}$$

For an augmented operad O (resp. a coaugmented cooperad C) the notation $O_{\circ}$ (resp. $C_{\circ}$) is reserved for the kernel (resp. the cokernel) of the augmentation (resp. the coaugmentation). For example,

$$\mathsf{coCom}_{\circ}(n) = \begin{cases} \mathbb{K} & \text{if} \;\; n \ge 2\,,\\ \mathbf{0} & \text{otherwise}\,. \end{cases}$$

We denote by Λ the endomorphism operad of the 1-dimensional vector space $\mathbf{s}^{-1}\mathbb{K}$ placed in degree -1

$$\Lambda = \mathrm{End}_{\mathbf{s}^{-1}\mathbb{K}}\,. \tag{1.8}$$

[1]In our paper, we often identify invariants and coinvariants using the fact that the underlying field $\mathbb{K}$ has characteristic zero.

In other words,

$$\Lambda(n) = \mathbf{s}^{1-n}\mathrm{sgn}_n\,,$$

where sgn_n is the sign representation for the symmetric group S_n We observe that the collection Λ is also naturally a cooperad.

For a dg operad (resp. a dg cooperad) P in we denote by ΛP the dg operad (resp. the dg cooperad) which is obtained from P via tensoring with Λ, i.e.

$$\Lambda P(n) = \mathbf{s}^{1-n} P(n) \otimes \mathrm{sgn}_n\,. \tag{1.9}$$

For example, an algebra over $\Lambda\mathsf{Lie}$ is a graded vector space V equipped with the binary operation:

$$\{\,,\,\} : V \otimes V \to V$$

of degree -1 satisfying the identities:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\}$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0\,,$$

where v_1, v_2, v_3 are homogeneous vectors in V.

$\mathsf{Ger}^\vee$ denotes the Koszul dual cooperad [26], [30], [31] for Ger. It is known [33] that

$$\mathsf{Ger}^\vee = \Lambda^2\mathsf{Ger}^*\,, \tag{1.10}$$

where Ger^* is obtained from the operad Ger by taking the linear dual. In other words, algebras over the linear dual $(\mathsf{Ger}^\vee)^*$ are very much like Gerstenhaber algebras except that the bracket carries degree 1 and the multiplication carries degree 2.

The notation Cobar is reserved for the cobar construction (see [21, Section 3.7]).

A graph Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite non-empty set and $E(\Gamma)$ is a set of unordered pairs of elements of $V(\Gamma)$. Elements of $V(\Gamma)$ are called vertices and elements of $E(\Gamma)$ are called edges. We say that a graph Γ is *labeled* if it is equipped with a bijection between the set $V(\Gamma)$ and the set of numbers $\{1, 2, \dots, |V(\Gamma)|\}$. We allow a graph with the empty set of edges. An *orientation* of Γ is a choice of directions on all edges of Γ.

In this paper, X denotes a smooth algebraic variety over $\mathbb{K}$. We denote by $\mathcal{O}_X$ the structure sheaf on X, by $\mathcal{T}_X$ (resp. $\mathcal{T}^*_X$) the tangent (resp. cotangent) sheaf and by

$$\mathcal{T}_{\mathrm{poly}} = S_{\mathcal{O}_X}(\mathbf{s}\mathcal{T}_X) \tag{1.11}$$

the sheaf of polyvector fields.

1.2 Trimming operadic algebras

Here we present a special construction which is used throughout the text.

Let $\mathcal{V}$ be an algebra over a (possibly colored) dg operad O with the underlying graded operad $\widetilde{O}$. Furthermore, let $\{i_{\mathfrak{v}}\}_{\mathfrak{v}\in\mathcal{S}}$ be a set of degree -1 derivations of the $\widetilde{O}$-algebra $\mathcal{V}$.

Let $\mathcal{V}^{[\mathcal{S}]}$ be the subcomplex of *basic elements* of $\mathcal{V}$ with respect to $\mathcal{S}$, i.e.

$$\mathcal{V}^{[\mathcal{S}]} := \{w \in \mathcal{V} \mid \forall\ \mathfrak{v} \in \mathcal{S} \quad i_{\mathfrak{v}}(w) = (\partial \circ i_{\mathfrak{v}} + i_{\mathfrak{v}} \circ \partial)(w) = 0\}\,, \tag{1.12}$$

where ∂ is the differential on $\mathcal{V}$. It is easy to see the O-algebra structure on $\mathcal{V}$ descends to the subcomplex of basic elements.

We call the construction of the O-algebra $\mathcal{V}^{[\mathcal{S}]}$ from an O-algebra $\mathcal{V}$ and a set of degree -1 derivations $\{i_{\mathfrak{v}}\}_{\mathfrak{v}\in\mathcal{S}}$ *trimming.*

Memorial note: Unfortunately, none of the authors met Boris Vasilievich Fedosov personally. However, we were influenced greatly by his works. We devote this paper in his memory.

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2 Formal coordinate systems and formal affine systems

Let X be a smooth algebraic variety of dimension d over an algebraically closed field $\mathbb{K}$ of characteristic zero.

In this section, we recall the construction of the sheaf $\mathcal{O}_X^{\mathrm{coord}}$ of formal coordinate systems and the sheaf $\mathcal{O}_X^{\mathrm{aff}}$ of formal affine systems mostly following [45].

We use these sheaves in Section 3 to construct the Fedosov resolution of the sheaf of tensor fields on X.

2.1 The sheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_X^{\mathbf{d}}$

Let R be a commutative $\mathbb{K}$-algebra with identity.

Let $\underline{i}, \underline{j}, \dots$ denote multi-indices $\underline{i} = (i_1, i_2, \dots, i_d)$, $\underline{j} = (j_1, j_2, \dots, j_d)$, with i_s, j_t being non-negative integers. For every multi-index $\underline{i}$ the notation $|\,\underline{i}\,|$ is reserved for the length of the multi-index $\underline{i}$, namely

$$|\,\underline{i}\,| = i_1 + i_2 + \cdots + i_d\,. \tag{2.1}$$

Definition 2.1 *Let us consider pairs*

$$(f, \underline{i}) \tag{2.2}$$

with $f \in R$ and $\underline{i}$ being a multi-index. We set $R^{\mathbf{d}}$ to be the quotient of the free commutative

algebra over R generated by pairs (2.2) with the respect to the ideal generated by relations[2]

$$
\begin{gathered}
((f+g),\underline{i}) = (f,\underline{i}) + (g,\underline{i})\,, \quad (fg,\underline{i}) = \sum_{\underline{j}+\underline{k}=\underline{i}} (f,\underline{j})(g,\underline{k})\,, \\
(f,(0,0,\ldots,0)) = f\,, \qquad (\lambda,\underline{i}) = 0 \text{ whenever } |\,\underline{i}\,| \geq 1
\end{gathered} \tag{2.3}
$$

$$\forall \quad f,g \in R, \quad \lambda \in \mathbb{K}\,.$$

It is convenient to use the notation $f_{\underline{i}}$ for the pair $(f,\underline{i})$. So, from now on, we switch to this notation.

It is also convenient to assign to each $f \in R$ the following formal Taylor power series

$$\widetilde{f} = \sum_{\underline{i}} f_{\underline{i}} t^{\underline{i}} \in R^{\mathbf{d}}[[t^1,t^2,\ldots,t^d]]\,, \tag{2.4}$$

where $t^{\underline{i}} = (t^1)^{i_1}(t^2)^{i_2}\ldots(t^d)^{i_d}$ and the summation goes over all multi-indices $\underline{i}$.

Using the notation $\widetilde{f}$ it is possible to rewrite the relations (2.3) in the following condensed form

$$\widetilde{(f+g)} = \widetilde{f} + \widetilde{g}\,, \qquad \widetilde{fg} = \widetilde{f}\,\widetilde{g}\,, \qquad \widetilde{f}\,\Big|_{t=0} = f\,, \qquad \widetilde{\lambda} = \lambda, \tag{2.5}$$

with $f,g \in R$ and $\lambda \in \mathbb{K}$.

Relations (2.5) imply that the formula

$$I(f) = \widetilde{f} \tag{2.6}$$

defines an injective homomorphism of $\mathbb{K}$-algebras from R to $R^{\mathbf{d}}[[t^1,t^2,\ldots,t^d]]$.

The construction of the R-algebra $R^{\mathbf{d}}$ from a $\mathbb{K}$-algebra R is functorial in the following sense: for every map of $\mathbb{K}$-algebras $\varphi : R \to \widetilde{R}$ we have a map (of $\mathbb{K}$-algebras)

$$\varphi^{\mathbf{d}} : R^{\mathbf{d}} \to \widetilde{R}^{\mathbf{d}}$$

which makes the following diagram commutative.

$$
\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d] & \widetilde{R} \arrow[d] \\
R^{\mathbf{d}} \arrow[r, "\varphi^{\mathbf{d}}"] & \widetilde{R}^{\mathbf{d}}
\end{tikzcd} \tag{2.7}
$$

Let f be a non-zero element of R. Applying the functoriality (2.7) to the natural map from R to its localization R_f we get the commutative diagram

$$
\begin{tikzcd}
R \arrow[r] \arrow[d] & R_f \arrow[d] \\
R^{\mathbf{d}} \arrow[r] & (R_f)^{\mathbf{d}}
\end{tikzcd} \tag{2.8}
$$

[2]Here we use the obvious structure of Abelian semigroup on the set of multi-indices.

of maps of $\mathbb{K}$-algebras.

Using the lower horizontal arrow in (2.8) we produce the obvious map of R_f-modules:

$$\psi_f : \left(R^{\mathbf{d}}\right)_f \to (R_f)^{\mathbf{d}} . \tag{2.9}$$

We claim that

Proposition 2.2 *For every commutative $\mathbb{K}$-algebra R and any non-zero element $f \in R$ the map ψ_f (2.9) is an isomorphism of R_f-modules.*

Proof. Since f is invertible in $\left(R^{\mathbf{d}}\right)_f$, the element $\widetilde{f}$ is invertible in the algebra $\left(R^{\mathbf{d}}\right)_f[[t^1, \dots, t^d]]$. Let us denote by $f^*_{\underline{i}} \in \left(R^{\mathbf{d}}\right)_f$ the coefficient in front of $t^{\underline{i}}$ in the series $(\,\widetilde{f}\,)^{-1} \in \left(R^{\mathbf{d}}\right)_f[[t^1, \dots, t^d]]$. For example,

$$f^*_{(0,0,\dots,0)} = \frac{1}{f} \,,$$

and

$$f^*_{(1,0,\dots,0)} = -\frac{f_{(1,0,\dots,0)}}{f^2} \,.$$

Next, we claim that the formulas

$$\nu_f(a_{\underline{i}}) := a_{\underline{i}} \,, \qquad \nu_f\big((f^{-1})_{\underline{i}}\big) := f^*_{\underline{i}} \,, \qquad a \in R \tag{2.10}$$

define a homomorphism of R_f-algebras

$$\nu_f : \left(R_f\right)^{\mathbf{d}} \to \left(R^{\mathbf{d}}\right)_f \,.$$

Indeed, formulas (2.10) define ν_f on generators of $\left(R_f\right)^{\mathbf{d}}$ and it is not hard to check ν_f respects all the defining relations.

Furthermore it is not hard to see that ν_f is the inverse of ψ_f (2.9). $\square$

Proposition 2.2 implies the following.

Corollary 2.3 *Let X be an algebraic variety over $\mathbb{K}$ and U be an affine open subset of X with $R_U = \mathcal{O}_X(U)$. Furthermore, let $(R^{\mathbf{d}}_U)\tilde{}$ be the quasicoherent sheaf on U corresponding to the R_U-module $R^{\mathbf{d}}_U$. Then the formula*

$$\mathcal{O}^{\mathbf{d}}_X \Big|_U := (R^{\mathbf{d}}_U)\tilde{}$$

defines a quasicoherent sheaf of $\mathcal{O}_X$-algebras. $\square$

Let us remark that the definition of the sheaf $\mathcal{O}^{\mathbf{d}}_X$ makes perfect sense for an arbitrary (not necessarily smooth) algebraic variety X.

For a smooth algebraic variety X we have the following statement.

Proposition 2.4 *Let X be a smooth algebraic variety over $\mathbb{K}$ of dimension d. Furthermore, let U be an affine subset of X which admits a global system of parameters:*

$$x^1, x^2, \dots, x^d \in \mathcal{O}_X(U) \,. \tag{2.11}$$

Then $\mathcal{O}^{\mathbf{d}}_X(U)$ is isomorphic to the polynomial algebra over $\mathcal{O}_X(U)$ in the generators

$$\big\{x^a_{\underline{i}}\big\}_{|\underline{i}|\ge 1,\ 1\le a\le d} \,. \tag{2.12}$$

Proof. Let us set

$$R := \mathcal{O}_X(U)\,.$$

Our goal is to show that the obvious map of commutative R-algebras

$$\varrho : R\big[\{x^a_{\underline{i}}\}_{|\underline{i}|\ge 1,\ 1\le a\le d}\big] \to R^{\mathbf{d}} \tag{2.13}$$

is an isomorphism.

For this purpose we introduce increasing filtrations on both algebras $R\big[\{x^a_{\underline{i}}\}_{|\underline{i}|\ge 1,\ 1\le a\le d}\big]$ and $R^{\mathbf{d}}$.

$$R = F^0 \subset F^1 \subset F^2 \subset F^3 \subset \dots \quad \subset R\big[\{x^a_{\underline{i}}\}_{|\underline{i}|\ge 1,\ 1\le a\le d}\big]\,,$$

$$R = F^0 R^{\mathbf{d}} \subset F^1 R^{\mathbf{d}} \subset F^2 R^{\mathbf{d}} \subset \dots \quad \subset R^{\mathbf{d}}\,,$$

where

$$F^m = R\big[\{x^a_{\underline{i}}\}_{1\le|\underline{i}|\le m,\ 1\le a\le d}\big]$$

and $F^m R^{\mathbf{d}}$ is the quotient of the polynomial algebra

$$R\big[\{f_{\underline{i}}\}_{f\in R,\ |\underline{i}|\le m}\big]$$

by the relations of $R^{\mathbf{d}}$ (2.3) involving only the elements $f_{\underline{i}}$ with $|\underline{i}| \le m$.

Furthermore, for each $m \ge 0$ we introduce the ideal $\widetilde{I}^m$ (resp. I^m) of the R-algebra $F^m R^{\mathbf{d}}$ (resp. F^m). The ideal $\widetilde{I}^m \subset F^m R^{\mathbf{d}}$ is generated by elements $f_{\underline{i}}$ where $f \in R$ and $1 \le |\underline{i}| \le m$. The ideal $I^m \subset F^m$ is generated by elements $x^a_{\underline{i}}$ for $1 \le a \le d$ and $|\underline{i}| \le m$. For $m = 0$ we set

$$I^0 = \widetilde{I}^0 = \mathbf{0}\,.$$

It is clear that the map ϱ (2.13) is compatible with the filtrations and moreover

$$\varrho(I^m) \subset \widetilde{I}^m\,.$$

Let us prove, by induction, that for each $m \ge 0$ the map ϱ (2.13) gives us an isomorphism from F^m to $F^m R^{\mathbf{d}}$ and

$$\varrho(I^m) = \widetilde{I}^m\,.$$

For $m = 0$ the statement is obvious. Let assume that the desired statement holds for $m-1$.

The R-algebra $F^m R^{\mathbf{d}}$ is obtained from $F^{m-1} R^{\mathbf{d}}$ via adjoining elements $f_{\underline{i}}$ with $|\underline{i}| = m$ and imposing the relations

$$\begin{gathered}(f+g)_{\underline{i}} = f_{\underline{i}} + g_{\underline{i}}\\ \lambda_{\underline{i}} = 0 \qquad \forall \quad \lambda \in \mathbb{K}\\ (fg)_{\underline{i}} = f_{\underline{i}} g + f g_{\underline{i}} + \dots\end{gathered} \tag{2.14}$$

where $\dots$ stands for a sum of elements in the ideal $\widetilde{I}^{m-1}$.

Therefore, the quotient R-algebra

$$F^m R^{\mathbf{d}} \big/ \widetilde{I}^{m-1} \tag{2.15}$$

is isomorphic to the symmetric algebra (over R) on $N(d,m)$-copies of the module $\Omega^1_{\mathbb{K}}(R)$ of Kähler differentials

$$S_R\Big(\,(\Omega^1_{\mathbb{K}}(R))^{\oplus\ N(d,m)}\,\Big)\,, \tag{2.16}$$

where $N(d,m)$ is the total number[3] of multi-indices $\underline{i}$ of length m.

Let us consider the following commutative diagram of maps of commutative rings:

$$\begin{tikzcd} 0 \arrow[r] & I^{m-1} \arrow[r] \arrow[d, "\varrho"] & F^m \arrow[r] \arrow[d, "\varrho"] & F^m/I^{m-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{I}^{m-1} \arrow[r] & F^m R^{\mathbf{d}} \arrow[r] & F^m R^{\mathbf{d}}/\widetilde{I}^{m-1} \arrow[r] & 0 \end{tikzcd} \tag{2.17}$$

Since R has a global system of parameters (2.11), $\Omega^1_{\mathbb{K}}(R)$ is a free[4] R-module on the 1-forms dx^a. Therefore, the right most vertical arrow in (2.17) is an isomorphism.

On the other hand, the left most vertical arrow is also an isomorphism by the induction assumption.

Thus, by the five lemma, the middle vertical arrow in (2.17) is also an isomorphism.

It remains to show that for every $f \in R$ and for every multi-index $\underline{i}$ of length m the element $f_{\underline{i}}$ belongs to $\varrho(I^m)$.

Since variables $x^1, x^2, \dots, x^d$ form a global system of parameters for R, for each $f \in R$ there exists a unique collection of elements $\{f_a\}_{1\le a\le d} \in R$ such that

$$df = \sum_{a=1}^{d} f_a\, dx^a\,.$$

Hence, using the above isomorphism between the quotient (2.15) and the symmetric algebra (2.16) we deduce that, for every multi-index $\underline{i}$ of length m,

$$f_{\underline{i}} = \sum_{a=1}^{d} f_a\, x^a_{\underline{i}} + \dots\,, \tag{2.18}$$

where $\dots$ stands for a sum of elements in the ideal $\widetilde{I}^{m-1}$.

Thus, due to the inductive assumption, $f_{\underline{i}}$ belongs to $\varrho(I^m)$.

Proposition 2.4 is proved. $\square$

From now on, we assume that X is a smooth algebraic variety over $\mathbb{K}$ of dimension d.

2.2 The sheaf of formal coordinate systems $\mathcal{O}_X^{\mathrm{coord}}$

In this section, we use the sheaf $\mathcal{O}_X^{\mathbf{d}}$ to construct the sheaf of formal coordinate systems $\mathcal{O}_X^{\mathrm{coord}}$ for any smooth algebraic variety X over $\mathbb{K}$.

[3] $N(d,m)$ is also the number of integer points inside the $(d-1)$-simplex $\{(u_1, u_2, \dots, u_d),\ \ u_i \ge 0,\ \ u_1 + u_2 + \dots + u_d = m\}$.

[4] This is an easy exercise from algebraic geometry.

Let R be the ring of functions $\mathcal{O}(U)$ on a smooth affine variety U of dimension d. Let us assume that U has a global system of parameters

$$x^1, x^2, \ldots, x^d \in R\,. \tag{2.19}$$

For every x^a we rewrite the formal series $\widetilde{x^a} \in R^{\mathbf{d}}[[t^1, \ldots, t^d]]$ as follows

$$\widetilde{x^a} = x^a + \sum_{b=1}^{d} x^a_{(b)} t^b + \sum_{\underline{i},\ |\underline{i}|>1} x^a_{\underline{i}} t^{\underline{i}} \tag{2.20}$$

where (b) denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots 0)$ with 1 placed in the b-th spot and $|\,\underline{i}\,|$ is the length (2.1) of the multi-index $\underline{i}$.

Proposition-Definition 2.5 *We define the $\mathbb{K}$-algebra $R^{\rm coord}$ as the localization of $R^{\mathbf{d}}$ with respect to the element*

$$\det ||x^a_{(b)}||\,.$$

This definition does not depend on the choice of the system of regular parameters (2.19).

Proof. Since $x^1, x^2, \ldots, x^d \in R$ form a global system of parameters on U the module $\Omega^1_{\mathbb{K}}(R)$ of Kähler differentials is freely generated by $\{dx^a\}_{1\le a\le d}$. In particular, for every $f \in R$ the element df can be written uniquely in the form

$$df = f_a\, dx^a\,, \qquad f_a \in R\,. \tag{2.21}$$

Let $y^1, \ldots, y^d \in R$ be another global system of parameters. The above observation implies that there exist elements $\Lambda^a_b \in R$ such that

$$dy^a = \Lambda^a_b\, dx^b\,. \tag{2.22}$$

Furthermore, since the elements $y^1, \ldots, y^d$ also form a global system of parameters, the R-valued matrix $||\Lambda^a_b||$ has to be invertible.

In order to prove the proposition, we observe that the operation

$$\mathfrak{d}_b(f) = f_{(b)} : R \to R^{\mathbf{d}} \tag{2.23}$$

is a $\mathbb{K}$-linear derivation of R-modules.

Therefore, for every $f \in R$ we have

$$\mathfrak{d}_b(f) = f_a\, x^a_{(b)}\,, \tag{2.24}$$

where $f_a \in R$ are the coefficients in the decomposition (2.21).

Thus, for y^a we have

$$y^a_{(b)} = \Lambda^a_c\, x^c_{(b)} \tag{2.25}$$

and hence

$$\det ||y^a_{(b)}|| = \det ||\Lambda^a_b||\, \det ||x^a_{(b)}||\,.$$

The desired statement follows immediately from the fact that the matrix $||\Lambda^a_b||$ is invertible. $\square$

Let X be an arbitrary smooth algebraic variety over $\mathbb{K}$ and U be an affine open subset of X equipped with a global system of parameters. Furthermore, let $R_U = \mathcal{O}_X(U)$ and $\left(R_U^{\mathrm{coord}}\right)^{\sim}$ be the quasi-coherent sheaf on U corresponding to the R_U-module R_U^{coord}. Combining Corollary 2.3 with Proposition-Definition 2.5 we see that the formula

$$\mathcal{O}_X^{\mathrm{coord}}\Big|_U := \left(R_U^{\mathrm{coord}}\right)^{\sim}$$

defines a quasi-coherent sheaf $\mathcal{O}_X^{\mathrm{coord}}$ of $\mathcal{O}_X$-algebras over X. We call $\mathcal{O}_X^{\mathrm{coord}}$ *the sheaf of formal coordinate systems.*

Proposition 2.4 implies the following statement:

Corollary 2.6 *If X is a smooth algebraic variety over $\mathbb{K}$ of dimension d and U is an affine subset of X which admits a global system of parameters*

$$x^1, x^2, \ldots, x^d \in \mathcal{O}_X(U), \tag{2.26}$$

then $\mathcal{O}_X^{\mathrm{coord}}(U)$ is isomorphic to the quotient

$$\mathcal{O}_X(U)\left[\{x^a_{\underline{i}}\}_{|\underline{i}|\geq 1,\ 1\leq a\leq d} \cup \{K\}\right] \Big/ \langle K \det ||x^a_{(b)}|| - 1\rangle \tag{2.27}$$

of the polynomial algebra over $\mathcal{O}_X(U)$ in the variables

$$\{x^a_{\underline{i}}\}_{|\underline{i}|\geq 1,\ 1\leq a\leq d} \ \cup\ \{K\}$$

with respect to the ideal generated by the element $K \det ||x^a_{(b)}|| - 1$. $\square$

2.3 The sheaf of formal affine systems $\mathcal{O}_X^{\mathrm{aff}}$

Let us start by observing that there is an obvious bijection between the set of multi-indices

$$\{\underline{i} = (i_1, i_2, \ldots, i_d) \mid i_s \geq 0,\ |\,\underline{i}\,| \geq 1\}$$

and the set of symmetric multi-indices

$$\left\{(a_1, a_2, \ldots, a_k) \mid 1 \leq a_t \leq d,\ k \geq 1\right\} \Big/ (\ldots, a_s, \ldots a_t, \ldots) = (\ldots, a_t, \ldots a_s, \ldots)\,. \tag{2.28}$$

This bijection assigns to the multi-index $\underline{i}$ the symmetric multi-index

$$(\underbrace{1, 1, \ldots, 1}_{i_1 \text{ times}}, \underbrace{2, 2, \ldots, 2}_{i_2 \text{ times}}, \ldots, \underbrace{d, d, \ldots, d}_{i_d \text{ times}})\,. \tag{2.29}$$

Notice that k in (2.28) is exactly the length $|\,\underline{i}\,|$ of the multi-index $\underline{i}$.

For a symmetric multi-index $(a_1, a_2, \ldots, a_k)$ corresponding to $\underline{i} = (i_1, i_2, \ldots, i_d)$ and $f \in R$ we set[5]

$$f_{(a_1, a_2, \ldots, a_k)} := i_1!\, i_2! \,\ldots\, i_d!\ f_{\underline{i}}\,. \tag{2.30}$$

[5]Here R is the algebra of functions on a smooth affine variety U.

It is clear that the R-algebra $R^{\mathbf{d}}$ is the quotient of the free R-algebra in elements

$$\left\{f_{(a_1,a_2,\ldots,a_k)}\right\}_{1\le a_t\le d,\ k\ge 1} \tag{2.31}$$

with respect to the ideal generated by the relations

$$f_{(\ldots,a_i,a_{i+1},\ldots)} = f_{(\ldots,a_{i+1},a_i,\ldots)}\,, \qquad \lambda_{(a_1,a_2,\ldots,a_k)} = 0\,, \tag{2.32}$$

$$\widetilde{(f+g)} = \widetilde{f} + \widetilde{g}\,, \qquad \widetilde{fg} = \widetilde{f}\ \widetilde{g}\,, \tag{2.33}$$

where $\lambda \in \mathbb{K}$, $f, g \in R$ and

$$\widetilde{f} = f + \sum_{k\ge 1}\sum_{1\le a_t\le d}\frac{1}{k!} f_{(a_1,a_2,\ldots,a_k)} t^{a_1}t^{a_2}\ldots t^{a_k} \in R^{\mathbf{d}}[[t^1,t^2,\ldots,t^d]]\,. \tag{2.34}$$

Equation (2.30) allows us to switch back and forth between the sets of generators $\{f_{\underline{i}}\}_{|\underline{i}|\ge 1}$ and (2.31).

We claim that

Proposition 2.7 *The formula*

$$h(f_{(a_1,a_2,\ldots,a_k)}) = \sum_{1\le b_1,\ldots,b_k\le d} h_{a_1}^{b_1}h_{a_2}^{b_2}\ldots h_{a_k}^{b_k} f_{(b_1,b_2,\ldots,b_k)}\,, \tag{2.35}$$

$$h = ||h_a^b|| \in \mathrm{GL}_d(\mathbb{K})$$

defines a left action of the affine algebraic group $\mathrm{GL}_d(\mathbb{K})$ *on the* R*-algebra* $R^{\mathbf{d}}$ *This action extends in the obvious way to* R^{coord} *and to the sheaf of* $\mathcal{O}_X$*-algebras* $\mathcal{O}_X^{\mathrm{coord}}$ *for any smooth algebraic variety* X *over* $\mathbb{K}$*.*

Proof. A direct computation shows that for every pair $h, h' \in \mathrm{GL}_d(\mathbb{K})$

$$h\big(h'(f_{(a_1,a_2,\ldots,a_k)})\big) = hh'(f_{(a_1,a_2,\ldots,a_k)})\,.$$

It is also clear that the ideal generated by relations (2.32) and (2.33) is closed with the respect to the action of $\mathrm{GL}_d(\mathbb{K})$.

Thus, formula (2.35) indeed defines a left action of $\mathrm{GL}_d(\mathbb{K})$ on the R-algebra $R^{\mathbf{d}}$.

Let us recall that R^{coord} is obtained from $R^{\mathbf{d}}$ by localizing with respect to the element

$$\det ||x^a_{(b)}||\,,$$

where $x^1,\ldots,x^d$ is the global system of parameters for $\mathrm{Spec}(R)$.

Since for every $h \in \mathrm{GL}_d(\mathbb{K})$

$$h(x^a_{(b)}) = \sum_{c=1}^{d} h_b^c\, x^a_{(c)}$$

the action (2.35) of $\mathrm{GL}_d(\mathbb{K})$ extends to the localization R^{coord} of $R^{\mathbf{d}}$.

This action also obviously extends to the sheaf $\mathcal{O}_X^{\mathrm{coord}}$ of $\mathcal{O}_X$-algebras on any smooth algebraic variety X over $\mathbb{K}$. $\square$

Remark 2.8 We would like to mention that formula (2.35) makes sense if $h \in \mathrm{GL}_d(R^{\mathrm{coord}})$ and $f_{(a_1,a_2,\dots,a_k)}$ are considered as elements of R^{coord}. In other words, the action of $\mathrm{GL}_d(\mathbb{K})$ on R^{coord} extends to the action of $\mathrm{GL}_d(R^{\mathrm{coord}})$ on R^{coord}.

Using the action (2.35) of $\mathrm{GL}_d(\mathbb{K})$ on $\mathcal{O}_X^{\mathrm{coord}}$, we define the sheaf $\mathcal{O}_X^{\mathrm{aff}}$ of formal affine coordinate systems.

Definition 2.9 *Let X be a smooth algebraic variety over $\mathbb{K}$. The sheaf $\mathcal{O}_X^{\mathrm{aff}}$ of* formal affine coordinate systems *is the subsheaf of $\mathrm{GL}_d(\mathbb{K})$-invariant sections of $\mathcal{O}_X^{\mathrm{coord}}$.*

Since any section of the structure sheaf $\mathcal{O}_X$ is obviously invariant under the $\mathrm{GL}_d(\mathbb{K})$-action, the sheaf $\mathcal{O}_X^{\mathrm{aff}}$ is naturally a sheaf of $\mathcal{O}_X$-algebras.

Let us observe that the R-algebra $R^{\mathbf{d}}[[t^1,\dots t^d]]$ carries two left $\mathrm{GL}_d(\mathbb{K})$-actions: the first one is obtained by extending the action (2.35) by $R[[t^1,\dots t^d]]$-linearity; the second one is obtained by setting

$$\mathscr{A}_h(t^a) = \sum_{b=1}^{d} (h^{-1})^a_b\, t^b\,, \qquad \mathscr{A}_h(f_{\underline{i}}) = 0\,, \qquad \forall \ \ \underline{i}\,. \tag{2.36}$$

Remark 2.10 It is easy to see that the above actions of $\mathrm{GL}_d(\mathbb{K})$ on $R^{\mathbf{d}}[[t^1,\dots t^d]]$ commute and hence the formula

$$F \in R^{\mathbf{d}}[[t^1,\dots t^d]] \ \mapsto\ h \circ \mathscr{A}_h(F) \in R^{\mathbf{d}}[[t^1,\dots t^d]] \tag{2.37}$$

defines another left action on the R-algebra $R^{\mathbf{d}}[[t^1,\dots t^d]]$. Let us also remark that for every $f \in R$ and $h \in \mathrm{GL}_d(\mathbb{K})$ we have

$$h \circ \mathscr{A}_h(\ \widetilde{f}\ ) = \ \widetilde{f}\ . \tag{2.38}$$

Differentiating the actions (2.35), (2.36) of $\mathrm{GL}_d(\mathbb{K})$ on $R^{\mathbf{d}}$ and $R^{\mathbf{d}}[[t^1,\dots t^d]]$, respectively we obtain the corresponding actions of the Lie algebra $\mathfrak{gl}_d(\mathbb{K})$. The latter action is given by the assignment

$$\mathfrak{v} = ||\mathfrak{v}^a_b|| \ \mapsto\ \ -\,\mathfrak{v}^a_b t^b \frac{\partial}{\partial t^a} \in \mathrm{Der}_{R^{\mathbf{d}}}\big(R^{\mathbf{d}}[[t^1,\dots t^d]]\big) \tag{2.39}$$

and the former

$$\mathfrak{v} \mapsto \overline{\mathfrak{v}} \in \mathrm{Der}_R(R^{\mathbf{d}})$$

is defined by declaring that

$$\sum_{\underline{i}} \overline{\mathfrak{v}}(f_{\underline{i}}) t^{\underline{i}} + \mathfrak{v}(\widetilde{f}) = 0\,, \qquad \forall \ \ f \in R\,. \tag{2.40}$$

The actions (2.35), (2.36), and (2.37) of $\mathrm{GL}_d(\mathbb{K})$ on $R^{\mathbf{d}}[[t^1,\dots t^d]]$ extend in the obvious way to left actions on the R-algebra $R^{\mathrm{coord}}[[t^1,\dots t^d]]$. By abuse of notation, we will denote by $\overline{\mathfrak{v}}$ the R-derivation of R^{coord} corresponding to the action (2.35) of $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$.

To give a local description of the sheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_X^{\mathrm{aff}}$, we consider an affine subset $U \subset X$ which admits a global system of parameters

$$x^1, x^2, \dots, x^d \in \mathcal{O}_X(U)\,. \tag{2.41}$$

Let us denote by u_x the invertible $d\times d$-matrix with entries $x^a_{(b)}$:

$$u_x = ||x^a_{(b)}|| \in \mathrm{GL}_d(R^{\mathrm{coord}})\,. \tag{2.42}$$

It is obvious that the elements[6]

$$u_x^{-1}(x^a_{\underline{i}})\,, \qquad 1 \le a \le d, \qquad |\underline{i}| \ge 2 \tag{2.43}$$

are $\mathrm{GL}_d(\mathbb{K})$-invariant. In other words,

$$u_x^{-1}(x^a_{\underline{i}}) \subset \mathcal{O}_X^{\mathrm{aff}}(U)\,.$$

For the sheaf $\mathcal{O}_X^{\mathrm{aff}}$ we have:

Proposition 2.11 (Proposition 6.3.1, [45]) *Let X be a smooth algebraic variety over $\mathbb{K}$ of dimension d and let U be an affine subset of X which admits a global system of parameters* (2.41)*. Then the map*

$$\varrho^{\mathrm{aff}} : \mathcal{O}_X(U)\Big[\,\{y^a_{\underline{i}}\}_{|\underline{i}|\ge 2,\ 1\le a\le d}\Big] \to \mathcal{O}_X^{\mathrm{aff}}(U)$$

$$\varrho^{\mathrm{aff}}(y^a_{\underline{i}}) := u_x^{-1}(x^a_{\underline{i}}) \tag{2.44}$$

is an isomorphism of $\mathcal{O}_X(U)$-algebras. The sheaf $\mathcal{O}_X^{\mathrm{aff}}$ can be equivalently defined as the sheaf of $\mathfrak{gl}_d(\mathbb{K})$-invariant sections of $\mathcal{O}_X^{\mathrm{coord}}$.

Proof. Let $R = \mathcal{O}_X(U)$. Due to Corollary 2.6, the commutative algebra R^{coord} is isomorphic to the quotient

$$R\big[\{x^a_{\underline{i}}\}_{|\underline{i}|\ge 1,\ 1\le a\le d} \cup \{K\}\big]/\langle K \det ||x^a_{(b)}|| - 1\rangle \tag{2.45}$$

The group $\mathrm{GL}_d(\mathbb{K})$ acts on generators $x^a_{\underline{i}}$ according to formula (2.35) and K transforms as

$$K \mapsto K/\det(h)\,,$$

where $h \in \mathrm{GL}_d(\mathbb{K})$.

To describe the algebra $R^{\mathrm{aff}} = (R^{\mathrm{coord}})^{\mathrm{GL}_d(\mathbb{K})}$ we consider the following isomorphism of R-algebras

$$\sigma : R\Big[\{y^a_{\underline{i}}\}_{|\underline{i}|\ge 2,\ 1\le a\le d} \cup \{x^a_{(b)}\}_{1\le a,b\le d} \cup \{K\}\Big]\,/\,\langle K \det ||x^a_{(b)}|| - 1\rangle \to R^{\mathrm{coord}} \tag{2.46}$$

$$\sigma(y^a_{\underline{i}}) = u_x^{-1}(x^a_{\underline{i}}), \qquad \sigma(x^a_{(b)}) = x^a_{(b)}, \qquad \sigma(K) = K\,.$$

The group $\mathrm{GL}_d(\mathbb{K})$ acts on the generators $y^a_{\underline{i}}$, $x^a_{(b)}$, K in the following way

$$y^a_{\underline{i}} \mapsto y^a_{\underline{i}}\,, \qquad x^a_{(b)} \mapsto h^c_b\, x^a_{(c)}\,, \qquad K \mapsto K/\det(h)\,,$$

where $h \in \mathrm{GL}_d(\mathbb{K})$.

[6] Here we use Remark 2.8.

Thus $R^{\rm coord}$ is isomorphic to

$$\mathcal{O}(\mathrm{GL}_d(\mathbb{K})) \otimes_{\mathbb{K}} R\Big[\{y^a_{\underline{i}}\}_{|\underline{i}|\ge 2,\ 1\le a\le d}\Big]\,, \tag{2.47}$$

where $\mathcal{O}(\mathrm{GL}_d(\mathbb{K}))$ is the algebra of regular functions on the algebraic group $\mathrm{GL}_d(\mathbb{K})$ and the $\mathrm{GL}_d(\mathbb{K})$-action on (2.47) is given by right translations on $\mathrm{GL}_d(\mathbb{K})$.

Since $\big(\mathcal{O}(\mathrm{GL}_d(\mathbb{K}))\big)^{\mathrm{GL}_d(\mathbb{K})} = \mathbb{K}$, we immediately conclude that equation (2.44), indeed defines an isomorphism

$$R\Big[\{y^a_{\underline{i}}\}_{|\underline{i}|\ge 2,\ 1\le a\le d}\Big] \cong R^{\rm aff}\,. \tag{2.48}$$

It is also clear that $\big(\mathcal{O}(\mathrm{GL}_d(\mathbb{K}))\big)^{\mathfrak{gl}_d(\mathbb{K})} = \mathbb{K}$. Hence,

$$R^{\rm aff} = \Big(R^{\rm coord}\Big)^{\mathfrak{gl}_d(\mathbb{K})}\,.$$

Proposition 2.11 is proven. $\square$

2.4 The canonical flat connection on $\Omega^{\bullet}(\mathcal{O}_X^{\rm coord})[[t^1,\dots t^d]]$

Let us consider the algebra

$$\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord}) \tag{2.49}$$

of exterior forms of the sheaf $\mathcal{O}_X^{\rm coord}$ of $\mathbb{K}$-algebras. We denote by d the de Rham differential on $\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})$.

We claim that

Theorem 2.12 *There exists a unique (degree 1) $\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})$-linear continuous[7] derivation*

$$\omega : \Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})[[t^1,\dots t^d]] \to \Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})[[t^1,\dots t^d]]$$

such that

$$d\,\widetilde{f} + \omega(\,\widetilde{f}\,) = 0 \tag{2.50}$$

for all local sections f of the sheaf $\mathcal{O}_X$. In addition, we have

$$(d+\omega)^2 = 0\,. \tag{2.51}$$

Proof. A degree 1 continuous $\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})$-linear derivation

$$\omega : \Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})[[t^1,\dots, t^d]] \to \Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})[[t^1,\dots, t^d]]$$

is uniquely determined by a collection of 1-forms:

$$\omega^a \in \Gamma(X, \Omega^1(\mathcal{O}_X^{\rm coord}))[[t^1,\dots,t^d]]$$

via the equation

$$\omega = \sum_{a=1}^{d} \omega^a \frac{\partial}{\partial t^a}\,.$$

[7] $\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}_X^{\rm coord})[[t^1,\dots t^d]]$ carries the obvious t-adic topology.

We will first define ω^a in terms a local system of parameters. Next, we will prove equation (2.51) and equation (2.50). Finally, we will deduce that ω does not depend on the choice of the local system.

Let $U \subset X$ be an affine subset of X with a global system of parameters $\{x^1, \dots, x^d\}$ and let $R = \mathcal{O}_X(U)$.

Since the matrix

$$||x^a_{(b)}||$$

is invertible in $\mathrm{Mat}_d(R^{\mathrm{coord}})$, the matrix

$$J_x = \left|\left|\frac{\partial \widetilde{x^a}}{\partial t^b}\right|\right| \tag{2.52}$$

is invertible in $\mathrm{Mat}_d(R^{\mathrm{coord}}[[t^1, \dots, t^d]])$.

Using this observation, it is easy to see that

$$\omega^a = -(J_x^{-1})^a_b \sum_{\underline{i}} dx^b_{\underline{i}} t^{\underline{i}} \tag{2.53}$$

is the unique solution of the system of equations:

$$d\,\widetilde{x^a} + \omega(\,\widetilde{x^a}\,) = 0\,, \qquad 1 \le a \le d\,. \tag{2.54}$$

To prove (2.50) we observe that the map

$$(d + \omega\cdot) \circ I : R \to \Omega^1(R^{\mathrm{coord}})[[t^1, \dots, t^d]] \tag{2.55}$$

is a $\mathbb{K}$-linear derivation of R-modules where the R-module structure on the target of (2.55) is defined by the formula

$$a \cdot v = I(a)\,v\,, \qquad a \in R, \quad v \in \Omega^1(R^{\mathrm{coord}})[[t^1, \dots, t^d]]\,.$$

Therefore, for every $f \in \mathcal{O}_X(U)$ we have

$$d\,\widetilde{f} + \omega(\,\widetilde{f}\,) = I(f_a)\,(d\,\widetilde{x^a} + \omega(\,\widetilde{x^a}\,))$$

where $f_a \in R$ are the coefficients in the decomposition (2.21).

Thus equation (2.50) follows from (2.54).

To prove equation (2.51), we remark that it is equivalent to

$$d\,\omega^a + \omega^b \wedge \frac{\partial \omega^a}{\partial t^b} = 0\,. \tag{2.56}$$

The latter equation can be verified by a direct computation using the obvious identities:

$$d\,(J_x^{-1})^a_b = -(J_x^{-1})^a_{a_1} \frac{\partial}{\partial t^{b_1}} d(\,\widetilde{x^{a_1}}\,)\,(J_x^{-1})^{b_1}_b\,,$$

$$\frac{\partial}{\partial t^c}(J_x^{-1})^a_b = -(J_x^{-1})^a_{a_1} \frac{\partial^2\,\widetilde{x^{a_1}}}{\partial t^c \partial t^{b_1}}\,(J_x^{-1})^{b_1}_b$$

and the symmetry of the expression

$$\frac{\partial^2 \widetilde{x^a}}{\partial t^b \partial t^c}$$

in the indices b and c.

It remains to show that ω does not depend on the choice of the system of parameters. Let $\{y^1, \ldots, y^d\}$ be another local system of parameters. Equation (2.50) implies that

$$d\,\widetilde{y^a} + \omega(\,\widetilde{y^a}\,) = 0\,, \qquad 1 \le a \le d\,. \tag{2.57}$$

But this system has the unique solution:

$$\omega^a = -(J_y^{-1})^a_b \sum_{\underline{i}} dy^b_{\underline{i}} t^{\underline{i}}. \tag{2.58}$$

Thus the construction of ω does not depend on the choice of the local system of parameters and the theorem is proven. $\square$

Let R be a smooth affine $\mathbb{K}$-algebra. Recall that $\mathfrak{gl}_d(\mathbb{K})$ acts by R-derivations on $R^{\mathbf{d}}$ and hence by R-derivations on R^{coord}. As above, we denote by $\overline{\mathfrak{v}}$ the R-derivation of R^{coord} corresponding to the action (2.35) of $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$. Furthermore, we denote by $i_{\overline{\mathfrak{v}}}$ the corresponding contraction operator on $\Omega^{\bullet}_{\mathbb{K}}(R^{\mathrm{coord}})$.

Using Theorem 2.12 we deduce the following.

Corollary 2.13 *Let X be a smooth algebraic variety over $\mathbb{K}$ of dimension d and ω be the derivation of $\Omega^{\bullet}_{\mathbb{K}}(\mathcal{O}^{\mathrm{coord}}_X)[[t^1, \ldots, t^d]]$ from Theorem 2.12. Then for every $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$ we have:*

$$i_{\overline{\mathfrak{v}}}\,\omega = -\mathfrak{v}^a_b t^b \frac{\partial}{\partial t^a}\,. \tag{2.59}$$

Proof. Using equations (2.39), (2.40) and (2.53) we deduce

$$\begin{aligned} i_{\overline{\mathfrak{v}}}(\omega^a) &= -(J_x^{-1})^a_b \sum_{\underline{i}} \overline{\mathfrak{v}}(x^b_{\underline{i}}) t^{\underline{i}} = -(J_x^{-1})^a_b\, \mathfrak{v}^c_{c'} t^{c'} \frac{\partial}{\partial t^c} \sum_{\underline{i}} x^b_{\underline{i}} t^{\underline{i}} \\ &= -\mathfrak{v}^c_{c'} t^{c'} (J_x^{-1})^a_b \frac{\partial\,\widetilde{x^b}}{\partial t^c} = -\mathfrak{v}^c_{c'} t^{c'} \delta^a_c = -\mathfrak{v}^a_{c'} t^{c'}\,. \end{aligned}$$

Hence

$$i_{\overline{\mathfrak{v}}}(\omega^a) \frac{\partial}{\partial t^a} = -\mathfrak{v}^a_b t^b \frac{\partial}{\partial t^a}$$

and the corollary is proven. $\square$

3 The Fedosov resolution of the tensor algebra of a smooth variety

In this section we present the Fedosov resolution of the sheaf of tensor fields on a smooth algebraic variety over an arbitrary algebraically closed field $\mathbb{K}$ of characteristic zero.

Recall that $\mathcal{T}_X$ (resp. $\mathcal{T}_X^*$) denotes the tangent (resp. cotangent) sheaf on a smooth algebraic variety X. We denote by $\mathcal{T}_X^{p,q}$ the sheaf of p-contravariant and q-covariant tensor fields on X, i.e.

$$\mathcal{T}_X^{p,q} := \underbrace{\mathcal{T}_X \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{T}_X}_{p \text{ times}} \otimes_{\mathcal{O}_X} \underbrace{\mathcal{T}_X^* \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{T}_X^*}_{q \text{ times}}\,. \tag{3.1}$$

For example, $\mathcal{T}_X^{0,0} = \mathcal{O}_X$; $\mathcal{T}_X^{1,0}$ (resp. $\mathcal{T}_X^{0,1}$) is the tangent (resp. cotangent) sheaf on X; and $\mathcal{T}_X^{1,1}$ is the sheaf of endomorphisms of the tangent sheaf $\mathcal{T}_X$ on X.

Under all possible contraction operations, the tensor product, and the action of the group $S_p \times S_q$, the collection of sheaves (3.1) carry an algebraic structure. This algebraic structure is governed by a colored[8] operad which we denote by $\mathfrak{T}$. We call the $\mathfrak{T}$-algebra

$$\left\{ \mathcal{T}_X^{p,q} \right\}_{p,q \ge 0} \tag{3.2}$$

the tensor algebra of X.

The goal of this section is to construct the Fedosov resolution of the tensor algebra for an arbitrary smooth algebraic variety X of dimension d.

For this purpose, we let $P = \mathbb{K}[[t^1, \dots, t^d]]$ be the (topological) algebra of formal Taylor power series in auxiliary variables $t^1, \dots, t^d$.

Next, we set

$$T^{p,q} := \underbrace{\mathrm{Der}_{\mathbb{K}}(P) \hat{\otimes}_P \dots \hat{\otimes}_P \mathrm{Der}_{\mathbb{K}}(P)}_{p \text{ times}} \hat{\otimes}_P \underbrace{\Omega^1_{\mathbb{K}}(P) \hat{\otimes}_P \dots \hat{\otimes}_P \Omega^1_{\mathbb{K}}(P)}_{q \text{ times}}, \tag{3.3}$$

where $\mathrm{Der}_{\mathbb{K}}(P)$ is the P-module of continuous derivations of P and

$$\Omega^1_{\mathbb{K}}(P) = \mathrm{Hom}_P(\mathrm{Der}_{\mathbb{K}}(P), P)\,.$$

In other words, elements of $T^{p,q}$ have the form

$$v = \sum_{1 \le a_t, b_s \le d} v^{a_1 \dots a_p}_{b_1 \dots b_q}\, \partial_{t^{a_1}} \otimes \cdots \otimes \partial_{t^{a_p}} \otimes dt^{b_1} \otimes \cdots \otimes dt^{b_q}\,,$$

where the components $v^{a_1 \dots a_p}_{b_1 \dots b_q} \in P$.

For a derivation

$$w = w^c(t) \partial_{t^c} \in \mathrm{Der}_{\mathbb{K}}(P)$$

and an element $v \in T^{p,q}$ we denote by $L_w(v)$ the Lie derivative of v along w. Recall that $L_w(v)$ has the following components:

$$L_w(v)^{a_1 \dots a_p}_{b_1 \dots b_q} = \sum_{c=1}^{d} w^c(t)\, \partial_{t^c} v^{a_1 \dots a_p}_{b_1 \dots b_q}(t) - \sum_{c=1}^{d} \sum_{i=1}^{p} (\partial_{t^c} w^{a_i}(t))\, v^{a_1 \dots a_{i-1}\, c\, a_{i+1} \dots a_p}_{b_1 \dots b_q}(t) \tag{3.4}$$

$$+ \sum_{c=1}^{d} \sum_{j=1}^{q} (\partial_{t^{b_j}} w^c(t))\, v^{a_1 \dots a_p}_{b_1 \dots b_{j-1}\, c\, b_{j+1} \dots b_q}(t)\,.$$

[8] The set of colors for $\mathfrak{T}$ is the set of pairs of non-negative integers (p, q).

It is clear that the collection

$$\Big\{ T^{p,q} \Big\}_{p,q \ge 0} \tag{3.5}$$

forms a $\mathfrak{T}$-algebra and L_w is a derivation of the $\mathfrak{T}$-algebra (3.5) for all $w \in \mathrm{Der}_{\mathbb{K}}(P)$.

Let ω be the global section of $\Omega^1(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} \mathrm{Der}_{\mathbb{K}}(P)$ defined in (2.53). Due to Theorem 2.12, the sum

$$d + L_\omega \tag{3.6}$$

is a differential on the sheaf of graded vector spaces

$$\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \tag{3.7}$$

for every pair $p, q \ge 0$. (The $\mathbb{Z}$-grading on (3.7) comes from the exterior degree on $\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})$.)

Since L_w is a derivation of the $\mathfrak{T}$-algebra (3.5) for every $w \in \mathrm{Der}_{\mathbb{K}}(P)$, the collection

$$\big\{ \Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \big\}_{p,q \ge 0} \tag{3.8}$$

together with the differential $d + L_\omega$ assembles into a sheaf of dg algebras over $\mathfrak{T}$.

For every $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$ the contraction $i_{\overline{\mathfrak{v}}}$ defines a degree -1 derivation of the sheaf of $\mathfrak{T}$-algebras (3.8). Furthermore, Corollary 2.13 implies that

$$[(d + L_\omega), i_{\overline{\mathfrak{v}}}] = l_{\overline{\mathfrak{v}}} - L_{\mathfrak{v}^a_b t^b \partial_{t^a}} \,, \tag{3.9}$$

where $l_{\overline{\mathfrak{v}}}$ denotes the action (2.35) of $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$ on $\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})$.

Due to Proposition 2.7 and Remark 2.10, the assignment

$$\mathfrak{v} = ||\mathfrak{v}^a_b|| \; \mapsto \; l_{\overline{\mathfrak{v}}} - L_{\mathfrak{v}^a_b t^b \partial_{t^a}}$$

defines an action on $\mathfrak{gl}_d(\mathbb{K})$ on the $\mathfrak{T}$-algebra (3.8). Moreover, due to equation (3.9), this action is compatible with the differential (3.6).

Let us construct a map of sheaves of $\mathfrak{T}$-algebras

$$\tau : \mathcal{T}_X^{p,q} \to \big(\mathcal{O}_X^{\mathrm{coord}} \hat{\otimes} T^{p,q} \big)^{\mathfrak{gl}_d(\mathbb{K})} \,. \tag{3.10}$$

For this purpose we consider an affine subset $U \subset X$ which admits a system of parameters

$$x^1, x^2, \dots, x^d \in R = \mathcal{O}_X(U) \,. \tag{3.11}$$

Furthermore, we denote by

$$\partial_{x^1}, \partial_{x^2}, \dots, \partial_{x^d} \tag{3.12}$$

the basis of derivations of R which is dual to the basis of Kähler differentials

$$dx^1, dx^2, \dots, dx^d \,. \tag{3.13}$$

Every tensor field $v \in \mathcal{T}_X^{p,q}(U)$ can be uniquely written in the form

$$v = \sum_{1 \le a_t, b_s \le d} v^{a_1 \dots a_p}_{b_1 \dots b_q} \, \partial_{x^{a_1}} \otimes \dots \otimes \partial_{x^{a_p}} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_q} \,, \tag{3.14}$$

where $v^{a_1\ldots a_p}_{b_1\ldots b_q} \in R$.

Next we set[9]

$$\tau(v) := I(v^{a_1\ldots a_p}_{b_1\ldots b_q})\,(J_x^{-1})^{a'_1}_{a_1}\ldots(J_x^{-1})^{a'_p}_{a_p}\,(J_x)^{b_1}_{b'_1}\ldots(J_x)^{b_q}_{b'_q}\partial_{t^{a'_1}}\otimes\cdots\otimes\partial_{t^{a'_p}}\otimes dt^{b'_1}\otimes\cdots\otimes dt^{b'_q}\,, \quad (3.15)$$

where I is defined in (2.6) and J_x is defined in (2.52).

We claim that

Claim 3.1 *The right hand side of (3.15) does not depend on the choice of the system of parameters (3.11). For every local section v of the sheaf $\mathcal{T}^{p,q}_X$ the image $\tau(v)$ is $\mathfrak{gl}_d(\mathbb{K})$-invariant and $(d+L_\omega)$-closed.*

Proof.

Let

$$\{y^1, y^2, \ldots, y^d\}$$

be another system of parameters on U and let Λ be the invertible $d\times d$ matrix with entries in R from equation (2.22).

Then components ${}^*v^{a_1\ldots a_p}_{b_1\ldots b_q}$ of v in the new basis

$$\Big\{\partial_{y^{a_1}}\otimes\cdots\otimes\partial_{y^{a_p}}\otimes dy^{b_1}\otimes\cdots\otimes dy^{b_q}\Big\}_{1\le a_t, b_s\le d}$$

are related to the component $v^{a_1\ldots a_p}_{b_1\ldots b_q}$ via the formula:

$${}^*v^{a_1\ldots a_p}_{b_1\ldots b_q} = \Lambda^{a_1}_{a'_1}\ldots\Lambda^{a_p}_{a'_p}(\Lambda^{-1})^{b'_1}_{b_1}\ldots(\Lambda^{-1})^{b_q}_{b'_q}\; v^{a'_1\ldots a'_p}_{b'_1\ldots b'_q}\,. \quad (3.16)$$

On the other hand, since I (2.6) is a $\mathbb{K}$-algebra homomorphism, we get

$$I({}^*v^{a_1\ldots a_p}_{b_1\ldots b_q}) = I(\Lambda^{a_1}_{a'_1})\ldots I(\Lambda^{a_p}_{a'_p})I\big((\Lambda^{-1})^{b'_1}_{b_1}\big)\ldots I\big((\Lambda^{-1})^{b'_q}_{b_q}\big)\; I(v^{a'_1\ldots a'_p}_{b'_1\ldots b'_q})\,.$$

So the proof of independence of the right hand side of (3.15) on the choice of the system of parameters boils down to checking the equation

$$\frac{\partial\,\widetilde{y^a}}{\partial t^c} = I(\Lambda^a_b)\,\frac{\partial\,\widetilde{x^b}}{\partial t^c}\,. \quad (3.17)$$

To prove equation (3.17) we notice that for every $1\le c\le d$ the map

$$\frac{\partial}{\partial t^c}\circ I : R \to R^{\mathrm{coord}}[[t^1, t^2, \ldots, t^d]] \quad (3.18)$$

is a $\mathbb{K}$-linear derivation of R-modules, where the R-module structure on the target of (3.18) is define by the formula:

$$a\cdot v = I(a)\,v\,, \qquad a\in R, \quad v\in R^{\mathrm{coord}}[[t^1, t^2, \ldots, t^d]]\,. \quad (3.19)$$

[9] We assume the summation over repeated indices.

Hence equation (2.22) and the universality of Kähler differentials implies equation (3.17). Thus the right hand side of (3.15) is indeed independent on the choice of system of parameters.

To prove that $\tau(v)$ is $(d+L_\omega)$-closed for every section v of $\mathcal{T}_X^{p,q}$, we give the following obvious identities of the matrix J_x

$$\partial_{t^b} \sum_{\underline{i}} dx^a_{\underline{i}} t^{\underline{i}} = d\,(J_x)^a_b\,, \tag{3.20}$$

$$\partial_{t^c}\,(J_x^{-1})^a_b = -(J_x^{-1})^a_{a_1}\left(\partial_{t^c}(J_x)^{a_1}_{b_1}\right)(J_x^{-1})^{b_1}_b\,, \tag{3.21}$$

and

$$\partial_{t^b}(J_x)^a_c = \partial_{t^c}(J_x)^a_b\,. \tag{3.22}$$

Using identities (3.20), (3.21) and (3.22), it is easy to see that

$$(d+L_\omega)\left((J_x^{-1})^{a'}_a \partial_{t^{a'}}\right) = 0 \tag{3.23}$$

and

$$(d+L_\omega)\left((J_x)^b_{b'} dt^{b'}\right) = 0\,. \tag{3.24}$$

On the other hand

$$(d+L_\omega)\big(I(v^{a_1\dots a_p}_{b_1\dots b_q})\big) = d\big(I(v^{a_1\dots a_p}_{b_1\dots b_q})\big) + \omega\big(I(v^{a_1\dots a_p}_{b_1\dots b_q})\big) = 0$$

due to (2.50).

Thus $\tau(v)$ is indeed $(d+L_\omega)$-closed for every tensor field v.

Finally, the $\mathfrak{gl}_d(\mathbb{K})$-invariance of $\tau(v)$ is obvious from the defining equation.

□

Let us now consider $\mathfrak{gl}_d(\mathbb{K})$ as the set of degree -1 derivations $i_{\overline{\mathfrak{v}}}$, $\mathfrak{v}\in\mathfrak{gl}_d(\mathbb{K})$ of the sheaf of dg $\mathfrak{T}$-algebras (3.8) and apply trimming to the sheaf (3.8) following Section 1.2.

Namely, according to (1.12), local sections of the sheaf

$$\left(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{p,q}\right)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{3.25}$$

are local sections w of $\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{p,q}$ satisfying the conditions

$$i_{\overline{\mathfrak{v}}}(w) = 0\,, \tag{3.26}$$

and

$$\big((d+L_\omega)\circ i_{\overline{\mathfrak{v}}} + i_{\overline{\mathfrak{v}}}\circ(d+L_\omega)\big)(w) = 0\,. \tag{3.27}$$

For example, if w is a local section of

$$\Omega^0(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{p,q} = \mathcal{O}_X^{\mathrm{coord}}\hat{\otimes}T^{p,q}$$

then equation (3.26) holds automatically and hence, equation (3.9) implies that

$$\left(\Omega^0(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{p,q}\right)^{[\mathfrak{gl}_d(\mathbb{K})]} = \left(\Omega^0(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{p,q}\right)^{\mathfrak{gl}_d(\mathbb{K})}. \tag{3.28}$$

Warning 3.2 We would like to recall that the notation $\mathcal{V}^{[\mathfrak{gl}_d(\mathbb{K})]}$ is reserved for the dg subalgebra of $\mathfrak{gl}_d(\mathbb{K})$-basic elements in $\mathcal{V}$ (see eq. (1.12)). On the other hand, we still use $\mathcal{V}^{\mathfrak{gl}_d(\mathbb{K})}$ to denote the subalgebra of $\mathfrak{gl}_d(\mathbb{K})$-invariants.

According to Section 1.2, the $\mathfrak{T}$-algebra structure descends to the collection

$$\left\{ \left(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \right\}_{p,q \ge 0} . \tag{3.29}$$

In other words, the collection (3.29) is a sheaf of dg algebras over the operad $\mathfrak{T}$.

Due to Claim 3.1 and equation (3.28), formula (3.15) defines a map of sheaves

$$\tau : \mathcal{T}_X^{p,q} \mapsto \left(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{3.30}$$

of dg algebras over $\mathfrak{T}$, where the sheaf $\mathcal{T}_X^{p,q}$ carries the zero differential.

Our goal is to show that (3.30) is a quasi-isomorphism of sheaves of of $\mathfrak{T}$-algebras.

Theorem 3.3 (Fedosov resolution of $\mathcal{T}_X^{p,q}$) *For every smooth algebraic variety X over $\mathbb{K}$ the map* (3.30) *defined by equation* (3.15) *is a quasi-isomorphism from the tensor algebra* (3.2) *on X to the dg $\mathfrak{T}$-algebra* (3.29)*.*

Proof. It is clear that the map τ intertwines all operations of the tensor algebra. Furthermore, by Claim 3.1, τ is compatible with the differential on

$$\left(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} .$$

Thus, it remains to prove that the complex of sheaves

$$\mathcal{T}_X^{p,q} \xrightarrow{\tau} \left(\Omega^0(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \left(\Omega^1(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \dots \tag{3.31}$$

is acyclic.

Since acyclicity of a complex of sheaves is a local property, it is enough to prove that the complex

$$\mathcal{T}_X^{p,q}(U) \xrightarrow{\tau} \left(\Omega^0(\mathcal{O}_X^{\mathrm{coord}})(U) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \left(\Omega^1(\mathcal{O}_X^{\mathrm{coord}})(U) \hat{\otimes} T^{p,q} \right)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \dots \tag{3.32}$$

is acyclic for "small enough" open neighborhood U of every point of X.

We will prove this fact for an arbitrary affine open subset $U \subset X$ which is equipped a global system of parameters

$$x^1, x^2, \dots, x^d \in \mathcal{O}_X(U) . \tag{3.33}$$

We set $R = \mathcal{O}_X(U)$ and observe that, for such an affine subset U, the complex in question is

$$T^{p,q}(R) \xrightarrow{\tau} \Big(\Omega^0(R^{\mathrm{coord}})\hat{\otimes} T^{p,q}\Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \Big(\Omega^1(R^{\mathrm{coord}})\hat{\otimes} T^{p,q}\Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \xrightarrow{d+L_\omega} \ \dots\,, \tag{3.34}$$

where

$$T^{p,q}(R) := \underbrace{\mathrm{Der}(R)\otimes_R \cdots \otimes_R \mathrm{Der}(R)}_{p \text{ times}} \otimes_R \underbrace{\Omega^1(R)\otimes_R \cdots \otimes_R \Omega^1(R)}_{q \text{ times}}\,. \tag{3.35}$$

We remark that the map τ (3.15) gives us the following isomorphism of $\mathbb{K}$-vector spaces

$$\Omega^k(R^{\mathrm{coord}}) \otimes T^{p,q} \cong \Omega^k(R^{\mathrm{coord}})[[t^1, t^2, \dots, t^d]] \otimes_{I(R)} \tau(T^{p,q}(R))\,. \tag{3.36}$$

Due to Proposition 2.11

$$R^{\mathrm{coord}} \cong R^{\mathrm{aff}} \otimes \mathcal{O}(\mathrm{GL}_d(\mathbb{K}))$$

on which $\mathrm{GL}_d(\mathbb{K})$ acts by right translations.

Hence,

$$\Big\{\eta \in \Omega^k(R^{\mathrm{coord}})[[t^1, t^2, \dots, t^d]] \ \Big|\ i_{\overline{\mathfrak{v}}}\eta = 0 \ \ \forall\, \mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})\Big\} \cong \tag{3.37}$$

$$\mathrm{Hom}_{R^{\mathrm{aff}}}\Big(\wedge^k_{R^{\mathrm{aff}}}\mathrm{Der}(R^{\mathrm{aff}}), R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]]\Big) \cong$$

$$\cong R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]] \otimes_{R^{\mathrm{aff}}} \Omega^k(R^{\mathrm{aff}})\,.$$

Thus, using (3.9), we get

$$\Big(\Omega^k(R^{\mathrm{coord}}) \otimes T^{p,q}\Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \cong$$

$$\big(R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]]\big)^{\mathfrak{gl}_d(\mathbb{K})} \otimes_{R^{\mathrm{aff}}} \Omega^k(R^{\mathrm{aff}}) \otimes_{I(R)} \tau(T^{p,q}(R))\,. \tag{3.38}$$

Therefore, since $\tau(T^{p,q}(R))$ is a free $I(R)$-module, it suffices to prove the acyclicity of the complex

$$R \xrightarrow{I} \Xi^0 \xrightarrow{(d+\omega)} \Xi^1 \xrightarrow{(d+\omega)} \Xi^2 \xrightarrow{(d+\omega)} \dots \tag{3.39}$$

with

$$\Xi^k = \big(R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]]\big)^{\mathfrak{gl}_d(\mathbb{K})} \otimes_{R^{\mathrm{aff}}} \Omega^k(R^{\mathrm{aff}}) \tag{3.40}$$

For this purpose we consider the continuous homomorphism of commutative R^{coord}-algebras

$$\psi : R^{\mathrm{coord}}[[\theta^1, \theta^2, \dots, \theta^d]] \to R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]] \tag{3.41}$$

defined by the equation

$$\psi(\theta^a) = I(x^a) - x^a\,. \tag{3.42}$$

Recall that

$$I(x^a) - x^a = x^a_{(b)} t^b + \sum_{\underline{i},\ |\underline{i}|\ge 2} x^a_{\underline{i}} t^{\underline{i}}$$

and the matrix $||x^a_{(b)}||$ is invertible in $\mathrm{Mat}_d(R^{\mathrm{coord}})$. Hence, the homomorphism ψ is an isomorphism.

Furthermore, since $I(x^a) - x^a$ is $\mathfrak{gl}_d(\mathbb{K})$-invariant for every a, the map ψ induces an isomorphism between the $\mathbb{K}$-algebras:

$$\Big(R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]]\Big)^{\mathfrak{gl}_d(\mathbb{K})} \cong R^{\mathrm{aff}}[[\theta^1, \theta^2, \dots, \theta^d]] \,. \tag{3.43}$$

On the other hand, by Theorem 2.12,

$$(d+\omega) I(x^a) = 0$$

and hence

$$(d+\omega)\psi(\theta^a) = -dx^a \,.$$

Thus, the cochain complex (3.39) is isomorphic to

$$R \xrightarrow{\psi\circ I} \Omega^0(R^{\mathrm{aff}})[[\theta^1, \theta^2, \dots, \theta^d]] \xrightarrow{D} \Omega^1(R^{\mathrm{aff}})[[\theta^1, \theta^2, \dots, \theta^d]] \xrightarrow{D} \tag{3.44}$$

where

$$D = d - \sum_{a=1}^{d} dx^a \frac{\partial}{\partial \theta^a} \,. \tag{3.45}$$

To deduce the acyclicity of the cochain complex (3.44), we need the following technical claim which is proved in Subsection 3.1 below.

Claim 3.4 *Let $U = \mathrm{Spec}(R)$ be a smooth affine variety over $\mathbb{K}$ of dimension d with a global system of parameters*

$$x^1, x^2, \dots, x^d \in R \,.$$

Let $\{\partial_{x^1}, \partial_{x^2}, \dots, \partial_{x^d}\}$ be the basis of $\mathrm{Der}(R)$ dual to $\{dx^1, dx^2, \dots, dx^d\}$. If

$$\psi : R^{\mathrm{coord}}[[\theta^1, \theta^2, \dots, \theta^d]] \to R^{\mathrm{coord}}[[t^1, t^2, \dots, t^d]]$$

is the isomorphism defined by (3.42), then for every $f \in R$

$$\psi \circ I(f) = f + \sum_{k\ge 1} \frac{1}{k!} \partial_{x^{a_1}} \partial_{x^{a_2}} \dots \partial_{x^{a_k}}(f) \theta^{a_1}\theta^{a_2}\dots\theta^{a_k} \,, \tag{3.46}$$

where indices $a_1, a_2, \dots, a_k$ run from 1 to d.

Due to Proposition 2.11

$$R^{\mathrm{aff}} \cong R[\{y^a_{\underline{i}}\}_{1\le a\le d,\, |\underline{i}|\ge 2}] \,.$$

Hence,

$$\Xi^k \cong \bigoplus_{p=0}^{k} \Omega^p(\mathbb{K}[\{y^a_{\underline{i}}\}_{1\le a\le d,\, |\underline{i}|\ge 2}]) \otimes \Omega^{k-p}(R)[[\theta^1, \theta^2, \dots, \theta^d]] \tag{3.47}$$

and the complex (3.44) is isomorphic to the tensor product of the de Rham complex

$$\big(\Omega^{\bullet}(\mathbb{K}[\{y^a_{\underline{i}}\}_{1\le a\le d,\, |\underline{i}|\ge 2}],\, d\big) \tag{3.48}$$

and the cochain complex

$$R \xrightarrow{\psi'} \Omega^0(R)[[\theta^1,\dots,\theta^d]] \xrightarrow{D'} \Omega^1(R)[[\theta^1,\dots,\theta^d]] \xrightarrow{D'} \tag{3.49}$$

where

$$\psi'(f) = f + \sum_{k\ge 1} \frac{1}{k!} \partial_{x^{a_1}} \partial_{x^{a_2}} \dots \partial_{x^{a_k}}(f) \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\,, \tag{3.50}$$

and

$$D' = d - \sum_{a=1}^{d} dx^a \frac{\partial}{\partial\theta^a}\,. \tag{3.51}$$

Let us prove that the cochain complex (3.49) is acyclic.

For this purpose we denote (3.49) by $\mathcal{K}$ and observe that it carries the descending filtration

$$\mathcal{K} = F_0\mathcal{K} \supset F_1\mathcal{K} \supset F_2\mathcal{K} \supset \dots \tag{3.52}$$

where $F_m\mathcal{K}$ consists of series

$$\sum_{p+q\ge m} f_{a_1,\dots a_p; b_1\dots b_q} dx^{a_1}\dots dx^{a_p}\theta^{b_1}\dots\theta^{b_q}\,, \qquad f_{a_1,\dots a_p; b_1\dots b_q} \in R\,.$$

The associated graded complex $\mathrm{Gr}(\mathcal{K})$ is isomorphic to

$$R \xrightarrow{\mathrm{Gr}(\psi')} \Omega^0(R)[\theta^1,\dots,\theta^d] \xrightarrow{\mathrm{Gr}(D')} \Omega^1(R)[\theta^1,\dots,\theta^d] \xrightarrow{\mathrm{Gr}(D')} \tag{3.53}$$

where

$$\mathrm{Gr}(\psi')(f) = f\,,$$

and

$$\mathrm{Gr}(D') = -\sum_{a=1}^{d} dx^a \frac{\partial}{\partial\theta^a}\,.$$

In other words, $\mathrm{Gr}(\mathcal{K})$ is isomorphic to the Koszul complex for the polynomial algebra $R[\theta^1,\dots,\theta^d]$ over R. Therefore $\mathrm{Gr}(\mathcal{K})$ is acyclic.

Combining this observation with the fact that $\mathcal{K}$ is complete with respect to filtration (3.52), we conclude that $\mathcal{K}$ is also acyclic.

Theorem 3.3 is proved. $\square$

3.1 Proof of Claim 3.4

The proof of Theorem 3.3 depends on Claim 3.4.

To prove this claim we observe that, since ψ intertwines the differentials $d+\omega$ and D (3.45), we conclude that

$$D(\psi \circ I(f)) = 0\,, \qquad \forall\ f \in R\,. \tag{3.54}$$

On the other hand,

$$D\left(f + \sum_{k\geq 1} \frac{1}{k!} \partial_{x^{a_1}} \partial_{x^{a_2}} \dots \partial_{x^{a_k}}(f) \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\right) = 0$$

by construction.

Thus we need to prove that, if a sum

$$\sum_{k\geq 1} \frac{1}{k!} C_{a_1 a_2 \dots a_k} \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\,, \qquad C_{a_1 a_2 \dots a_k} \in R^{\mathrm{aff}} \tag{3.55}$$

satisfies the equation

$$D\left(\sum_{k\geq 1} \frac{1}{k!} C_{a_1 a_2 \dots a_k} \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\right) = 0 \tag{3.56}$$

then the sum (3.55) is zero.

If (3.55) is non-zero then there exists a positive integer r such that at least one coefficient $C_{a_1 a_2 \dots a_r}$ is non-zero and all coefficients $C_{a_1 a_2 \dots a_k} = 0$ is $k < r$. Then,

$$D\left(\sum_{k\geq 1} \frac{1}{k!} C_{a_1 a_2 \dots a_k} \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\right) = \frac{1}{(r-1)!} C_{a_1 a_2 \dots a_r} dx^{a_1}\theta^{a_2}\dots\theta^{a_k} + \dots$$

where $\dots$ is a sum of terms of degree in θ greater than $r-1$.

Hence, if (3.55) is non-zero, then

$$D\left(\sum_{k\geq 1} \frac{1}{k!} C_{a_1 a_2 \dots a_k} \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}\right) \neq 0\,.$$

Thus $\psi \circ I(f)$ must equal

$$f + \sum_{k\geq 1} \frac{1}{k!} \partial_{x^{a_1}} \partial_{x^{a_2}} \dots \partial_{x^{a_k}}(f) \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}$$

for every $f \in R$.

Claim 3.4 is proved. $\square$

3.2 Fedosov resolution of the Gerstenhaber algebra of polyvector fields

Let us recall that antisymmetric contravariant tensor fields on X are called *polyvector fields*. In other words, polyvector fields are sections of the sheaf

$$\mathcal{T}_{\rm poly} := S_{\mathcal{O}_X}(\mathbf{s}\mathcal{T}_X)\,. \tag{3.57}$$

Let us also recall that the Schouten-Nijenhuis bracket $\{\ ,\ \}_{SN}$ and the obvious commutative multiplication equips $\mathcal{T}_{\rm poly}$ with the structure of a sheaf of Gerstenhaber algebras. We denote by Ger the Gerstenhaber operad.

Let us denote by $T_{\rm poly}(P)$ the Gerstenhaber algebra of poly-derivations of $P = \mathbb{K}[[t^1, t^2, \dots, t^d]]$, i.e.

$$T_{\rm poly}(P) := P\ \oplus\ \bigoplus_{m\ge 1}\Big(\underbrace{\mathbf{s}{\rm Der}(P)\hat{\otimes}_P \dots \hat{\otimes}_P\mathbf{s}{\rm Der}(P)}_{m \text{ times}}\Big)^{S_m} \tag{3.58}$$

Next consider the subsheaf

$$\Omega^{\bullet}(\mathcal{O}_X^{\rm coord})\hat{\otimes} T_{\rm poly}(P) \subset \bigoplus_{p\ge 0}\mathbf{s}^p\Omega^{\bullet}(\mathcal{O}_X^{\rm coord})\hat{\otimes}T^{p,0}. \tag{3.59}$$

It is easy to see that the subsheaf (3.59) is closed with respect to the differential (3.6). Furthermore, the restriction of the differential (3.6) to (3.59) coincides with the differential

$$d + \{\omega,\ \}_{SN}\,. \tag{3.60}$$

The sheaf (3.59) with the differential (3.60) is naturally a sheaf of dg Gerstenhaber algebras.

Since for every $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$, the operation $i_{\mathfrak{v}}$ is a derivation of the Gerstenhaber algebra structure on the sheaf (3.59), we may form[10] the following sheaf of dg Gerstenhaber algebras

$$\Big(\Omega^{\bullet}(\mathcal{O}_X^{\rm coord})\hat{\otimes} T_{\rm poly}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]}. \tag{3.61}$$

The sheaf of the dg Gerstenhaber algebras (3.61) with the differential (3.60) is a resolution of the sheaf (3.57) of polyvector fields on X. Namely,

Theorem 3.5 (Fedosov resolution for polyvector fields) *Let X be a smooth algebraic variety of dimension d over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let us consider the sheaf $\mathcal{T}_{\rm poly}$ (3.57) as a sheaf of dg Gerstenhaber algebras with the zero differential. Then the restriction of the map τ (3.15) to $\mathcal{T}_{\rm poly}$*

$$\tau\Big|_{\mathcal{T}_{\rm poly}}\ :\ \mathcal{T}_{\rm poly}\ \to\ \Big(\Omega^{\bullet}(\mathcal{O}_X^{\rm coord})\hat{\otimes} T_{\rm poly}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{3.62}$$

is a quasi-isomorphism of the sheaves of dg Gerstenhaber algebras.

[10] See Section 1.2.

Proof. Due to Theorem 3.3, the map

$$\tau : \mathcal{T}_X^{m,0} \to \Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{m,0}(P) \Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{3.63}$$

is a quasi-isomorphism of sheaves for every $m \geq 0$. Hence so is the map

$$\tau : \mathbf{s}^m\, \mathcal{T}_X^{m,0} \to \Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} \mathbf{s}^m\, T^{m,0}(P) \Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{3.64}$$

Let us denote by sgn_m the sign representation of S_m and observe that

$$\mathcal{T}_{\mathrm{poly}}^m = \Big(\mathrm{sgn}_m\, \mathbf{s}^m\, \mathcal{T}_X^{m,0} \Big)^{S_m}$$

and

$$\Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T_{\mathrm{poly}}^m(P) \Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \;=\; \bigg(\mathrm{sgn}_m \Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} \mathbf{s}^m\, T^{m,0}(P) \Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \bigg)^{S_m} .$$

Thus the map

$$\tau : \mathcal{T}_{\mathrm{poly}}^m \to \Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T_{\mathrm{poly}}^m(P) \Big)^{[\mathfrak{gl}_d(\mathbb{K})]}$$

is a quasi-isomorphism for every m, since, in characteristic zero, the cohomology commutes with taking invariants.

It remains to prove that the map (3.62) is compatible with the Gerstenhaber algebra structures.

To prove this property, we consider an affine open subset $U \subset X$, set $R = \mathcal{O}_X(U)$ and assume that U has a global system of parameters (3.11).

It is obvious from the definition of τ (3.15) that for every pair of sections $v_1, v_2 \in \mathcal{T}_{\mathrm{poly}}(U)$

$$\tau(v_1 \cdot v_2) = \tau(v_1) \cdot \tau(v_2) .$$

To prove the compatibility of τ with the bracket $\{\ ,\ \}_{SN}$, we observe that the Gerstenhaber algebra $\mathcal{T}_{\mathrm{poly}}(U)$ is generated by $f \in R$ and the derivations (3.12). Thus, it suffices to prove that

$$\{\tau(f_1), \tau(f_2)\}_{SN} = 0 , \tag{3.65}$$

$$\{\tau(\partial_{x^a}), \tau(f)\}_{SN} = \tau(\partial_{x^a}(f)) , \tag{3.66}$$

and

$$\{\tau(\partial_{x^a}), \tau(\partial_{x^b})\}_{SN} = 0 \tag{3.67}$$

for all $f, f_1, f_2 \in R$, and $1 \leq a, b \leq d$.

Since

$$\tau(f) = \widetilde{f} = f + \sum_{|\underline{i}| \geq 1} f_{\underline{i}} t^{\underline{i}}$$

equation (3.65) holds obviously.

To prove equation (3.66), we observe that the operation

$$f \mapsto \partial_{t^b}\, \widetilde{f} \, : R \to R^{\mathrm{coord}}[[t^1, \dots, t^d]]$$

is a $\mathbb{K}$-linear derivation of R-modules with $R^{\text{coord}}[[t^1, \dots, t^d]]$ carrying the R-module structure defined in (3.19).

Hence,

$$\partial_{t^b} \widetilde{f} = I(\partial_{x^a}(f)) \frac{\partial \widetilde{x^a}}{\partial t^b} . \tag{3.68}$$

Using equation (3.68) we deduce

$$\{\tau(\partial_{x^a}), \tau(f)\}_{SN} = (J_x^{-1})_a^b \partial_{t^b} \widetilde{f} = (J_x^{-1})_a^b I(\partial_{x^c}(f)) \frac{\partial \widetilde{x^c}}{\partial t^b} = I(\partial_{x^a}(f)) .$$

Thus equation (3.66) holds.

Using equations (3.21) and (3.22), it is easy to see that

$$\{\tau(\partial_{x^a}), \tau(\partial_{x^b})\}_{SN} = \Big((J_x^{-1})_a^c \, \partial_{t^c} (J_x^{-1})_b^{c'} - (J_x^{-1})_b^c \, \partial_{t^c} (J_x^{-1})_a^{c'} \Big) \partial_{t^{c'}} = 0 .$$

Thus equation (3.67) also holds.

Theorem 3.5 is proven. $\square$

4 Atiyah class via Fedosov resolution

Let us cover our variety X by affine open subsets $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ each of which has a global system of parameters:

$$x_\alpha^1, x_\alpha^2, \dots, x_\alpha^d \in \mathcal{O}_X(U_\alpha) . \tag{4.1}$$

For each $\alpha \in \mathcal{I}$, the module $\Omega^1(\mathcal{O}_X(U_\alpha))$ of Kähler differentials is freely generated by the forms

$$d\, x_\alpha^1, \;\; d\, x_\alpha^2, \;\; \dots, \;\; d\, x_\alpha^d . \tag{4.2}$$

Therefore, for every non-empty intersection[11] $U_{\alpha\beta} = U_\alpha \cap U_\beta$ there exists a unique non-degenerate matrix

$$||(\Lambda_{\alpha\beta})_b^a|| \in \mathrm{Mat}_d\big(\mathcal{O}_X(U_\alpha \cap U_\beta)\big)$$

such that

$$dx_\alpha^a = (\Lambda_{\alpha\beta})_b^a \, dx_\beta^b . \tag{4.3}$$

In particular, $\Lambda_{\beta\alpha} = (\Lambda_{\alpha\beta})^{-1}$.

It is easy to see that the collection of tensor fields

$$(\Lambda_{\beta\alpha})_a^{b_1} \, d(\Lambda_{\alpha\beta})_{b_2}^a \, \partial_{x_\beta^{b_1}} \otimes dx_\beta^{b_2} \in \Gamma(U_{\alpha\beta}, \mathcal{T}_X^{1,2}) \tag{4.4}$$

is a 1-cocycle in the Čech complex

$$\check{C}^\bullet(X, \mathcal{T}_X^{1,2}) \tag{4.5}$$

for the sheaf $\mathcal{T}_X^{1,2}$. Furthermore, the cohomology class of the cocycle (4.4) does not depend on the choice of the systems of parameters on affine subsets U_α .

[11] Let us recall that a variety is necessarily a separated scheme. Hence intersection of two affine subsets is again an affine subset.

According to [3], the cocycle (4.4) is trivial if and only if the tangent sheaf $\mathcal{T}_X$ admits an algebraic connection. We refer to the cohomology class of (4.4) as the *Atiyah class* of the tangent sheaf $\mathcal{T}_X$ and denote the cocycle (4.4) by A.

We claim that

Theorem 4.1 *Let $\omega^a \partial_{t^a}$ be the global section of the sheaf*

$$\Omega^1(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{1,0}(P)$$

introduced in Theorem 2.12. Then

$$\mathsf{A}_\omega := -\frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} \, \partial_{t^a} \otimes dt^{b_1} \otimes dt^{b_2} \tag{4.6}$$

is a global $(d + L_\omega)$-closed section of the sheaf

$$\big(\Omega^1(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{1,2}(P)\big)^{[\mathfrak{gl}_d(\mathbb{K})]} . \tag{4.7}$$

Furthermore, A_ω is cohomologous to the cocycle $\tau(\mathsf{A})$ in the Čech complex

$$\check{C}^{\bullet}\Big(X, \big(\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T^{1,2}(P)\big)^{[\mathfrak{gl}_d(\mathbb{K})]}\Big) . \tag{4.8}$$

Proof. To prove the first statement we need to show that A_ω satisfies these conditions

$$i_{\overline{\mathfrak{v}}}(\mathsf{A}_\omega) = 0 \,, \qquad [(d + L_\omega), i_{\overline{\mathfrak{v}}}](\mathsf{A}_\omega) = 0 \,, \qquad \forall \;\; \mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K}) \,, \tag{4.9}$$

and

$$(d + L_\omega)\mathsf{A}_\omega = 0 \,. \tag{4.10}$$

Applying $\partial_{t^{b_1}} \partial_{t^{b_2}}$ to equation (2.56) we get

$$0 = \partial_{t^{b_1}} \partial_{t^{b_2}} (d\omega^a + \omega^{a'} \partial_{t^{a'}}(\omega^a)) = d \, \frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} + \omega^{a'} \frac{\partial}{\partial t^{a'}} \left(\frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} \right) +$$
$$+ \frac{\partial^2 \omega^{a'}}{\partial t^{b_1} \partial t^{b_2}} \frac{\partial \omega^a}{\partial t^{a'}} + \frac{\partial \omega^{a'}}{\partial t^{b_1}} \frac{\partial^2 \omega^a}{\partial t^{a'} \partial t^{b_2}} + \frac{\partial \omega^{a'}}{\partial t^{b_2}} \frac{\partial^2 \omega^a}{\partial t^{a'} \partial t^{b_1}} = -\big(d\mathsf{A}_\omega + L_\omega(\mathsf{A}_\omega)\big)^a_{b_1 b_2} .$$

Thus equation (4.10) holds.

Due to Corollary 2.13, we have

$$i_{\overline{\mathfrak{v}}} \, \frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} = \frac{\partial^2}{\partial t^{b_1} \partial t^{b_2}} (i_{\overline{\mathfrak{v}}}(\omega^a)) = -\frac{\partial^2}{\partial t^{b_1} \partial t^{b_2}} (\mathfrak{v}^a_b t^b) = 0 \,.$$

Thus the first equation in (4.9) holds.

The second equation in (4.9) follows from the first one and (4.10).

Next, we observe that, due to equation (3.17),

$$\tau(\Lambda_{\beta\alpha}) = I(\Lambda_{\beta\alpha}) = J_{x_\beta} \, J^{-1}_{x_\alpha} \tag{4.11}$$

on every non-empty intersection $U_\alpha \cap U_\beta$.

Furthermore, since τ is compatible with the Schouten bracket, we have

$$\begin{aligned}\tau\big(dx^b_\beta \partial_{x^b_\beta}(\Lambda_{\alpha\beta})\big) &= dt^{b'} (J_\beta)^b_{b'}\, \tau(\partial_{x^b_\beta}\Lambda_{\alpha\beta}) = dt^{b'} (J_\beta)^b_{b'}\, \tau(\partial_{x^b_\beta})\big(\tau(\Lambda_{\alpha\beta})\big)\\ &= dt^{b'} (J_\beta)^b_{b'}\, (J^{-1}_\beta)^{b''}_b \frac{\partial}{\partial t^{b''}}\big(I(\Lambda_{\alpha\beta})\big) = dt^{b'} \frac{\partial}{\partial t^{b'}}\big(I(\Lambda_{\alpha\beta})\big)\\ &= dt^b \frac{\partial}{\partial t^b}(J_{x_\alpha} J^{-1}_{x_\beta})\,.\end{aligned}$$

Therefore

$$\begin{aligned}\tau(\mathsf{A}_{\alpha\beta}) &= \tau\Big((\Lambda_{\beta\alpha})^{b_1}_a\, d(\Lambda_{\alpha\beta})^a_{b_2}\, \partial_{x^{b_1}_\beta} \otimes dx^{b_2}_\beta\Big)\\ &= \left(J_{x_\beta} J^{-1}_{x_\alpha} dt^c \frac{\partial}{\partial t^c}(J_{x_\alpha} J^{-1}_{x_\beta})\right)^{b_1}_{\ \ b_2} (J^{-1}_{x_\beta})^{b'_1}_{b_1} (J_{x_\beta})^{b_2}_{b'_2} \partial_{t^{b'_1}} \otimes dt^{b'_2} \qquad (4.12)\\ &= dt^c\big(J^{-1}_{x_\alpha}\frac{\partial}{\partial t^c}(J_{x_\alpha})\big)^{b_1}_{\ \ b_2} \partial_{t^{b_1}} \otimes dt^{b_2} - dt^c\big(J^{-1}_{x_\beta}\frac{\partial}{\partial t^c}(J_{x_\beta})\big)^{b_1}_{\ \ b_2} \partial_{t^{b_1}} \otimes dt^{b_2}\,.\end{aligned}$$

It is not hard to see that for every affine chart U_α the section

$$dt^c\big(J^{-1}_{x_\alpha}\frac{\partial}{\partial t^c}(J_{x_\alpha})\big)^{b_1}_{\ \ b_2} \partial_{t^{b_1}} \otimes dt^{b_2}$$

is $\mathfrak{gl}_d(\mathbb{K})$-invariant.

Hence computation (4.12) implies that

$$\tau(\mathsf{A}) = -\check{\partial}(\mathsf{A}')\,, \qquad (4.13)$$

where A' is the Čech 0-cochain of

$$\Big(\Omega^0(\mathcal{O}^{\rm coord}_X)\hat{\otimes} T^{1,2}(P)\Big)^{\mathfrak{gl}_d(\mathbb{K})} = \Big(\Omega^0(\mathcal{O}^{\rm coord}_X)\hat{\otimes} T^{1,2}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]}$$

given by the equation

$$\mathsf{A}'_\alpha = dt^c\big(J^{-1}_{x_\alpha}\frac{\partial}{\partial t^c}(J_{x_\alpha})\big)^{b_1}_{\ \ b_2} \partial_{t^{b_1}} \otimes dt^{b_2}\,. \qquad (4.14)$$

Thus, equation (4.13) implies that $\tau(\mathsf{A})$ is cohomologous to the cocycle A'' with

$$\mathsf{A}''_\alpha = (d + L_\omega)\mathsf{A}'_\alpha \qquad (4.15)$$

For the components of $(d + L_\omega)\mathsf{A}'_\alpha$ we have

$$\begin{aligned}\Big((d+L_\omega)\mathsf{A}'_\alpha\Big)^a_{b_1 b_2} &= (J^{-1}_\alpha)^a_{a'} \frac{\partial^2}{\partial t^{b_1}\partial t^{b_2}}\, d\,\widetilde{x}^{\,a'} + \frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1}\partial t^{b_2}}\, d(J^{-1}_\alpha)^a_{a'}\\ &+ \omega^c \partial_{t^c}\left(\frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1}\partial t^{b_2}}\,(J^{-1}_\alpha)^a_{a'}\right) - \frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1}\partial t^{b_2}}\,(J^{-1}_\alpha)^c_{a'}\,\partial_{t^c}\omega^a\\ &+ \partial_{t^{b_1}}\omega^c \frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^c \partial t^{b_2}}\,(J^{-1}_\alpha)^a_{a'} + \partial_{t^{b_2}}\omega^c \frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1}\partial t^c}\,(J^{-1}_\alpha)^a_{a'}\,. \qquad (4.16)\end{aligned}$$

Using the equation $(d + \omega^c \partial_{t^c})\, \widetilde{x}^{\,a'} = 0$ and combining the first and the third term in the right hand side of (4.16) we get

$$
\begin{aligned}
&(J_\alpha^{-1})^a_{a'} \frac{\partial^2}{\partial t^{b_1} \partial t^{b_2}}\, d\, \widetilde{x}^{\,a'} + \omega^c \partial_{t^c} \left(\frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1} \partial t^{b_2}} (J_\alpha^{-1})^a_{a'} \right) \\
&= \omega^c \partial_{t^c} \left(\frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1} \partial t^{b_2}} (J_\alpha^{-1})^a_{a'} \right) - (J_\alpha^{-1})^a_{a'} \frac{\partial^2}{\partial t^{b_1} \partial t^{b_2}} \left(\omega^c \frac{\partial \widetilde{x}^{\,a'}}{\partial t^c} \right) \\
&= \begin{array}{l} -\dfrac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} - (\partial_{t^{b_1}} \omega^c) \dfrac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^c \partial t^{b_2}} (J_\alpha^{-1})^a_{a'} - (\partial_{t^{b_2}} \omega^c) \dfrac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1} \partial t^c} (J_\alpha^{-1})^a_{a'} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1} \partial t^{b_2}}\, \omega^c \partial_{t^c} (J_\alpha^{-1})^a_{a'}\,. \end{array}
\end{aligned} \tag{4.17}
$$

Therefore, we can rewrite equation (4.16) as follows:

$$
\begin{aligned}
\Big((d + L_\omega) \mathsf{A}'_\alpha \Big)^a_{b_1 b_2} &= - \frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}} \\
&\quad + \frac{\partial^2 \widetilde{x}^{\,a'}}{\partial t^{b_1} \partial t^{b_2}} \Big(d (J_\alpha^{-1})^a_{a'} + \omega^c \partial_{t^c} (J_\alpha^{-1})^a_{a'} - (J_\alpha^{-1})^c_{a'}\, \partial_{t^c} \omega^a \Big).
\end{aligned} \tag{4.18}
$$

Let us observe that the expression $d(J_\alpha^{-1})^a_{a'} + \omega^c \partial_{t^c} (J_\alpha^{-1})^a_{a'} - (J_\alpha^{-1})^c_{a'}\, \partial_{t^c} \omega^a$ is the a-th component of

$$
(d + L_\omega)\big((J_\alpha^{-1})^a_{a'} \partial_{t^a} \big)
$$

which is zero due to (3.23).

Thus

$$
\Big((d + L_\omega) \mathsf{A}'_\alpha \Big)^a_{b_1 b_2} = - \frac{\partial^2 \omega^a}{\partial t^{b_1} \partial t^{b_2}}
$$

and Theorem 4.1 follows. □

5 Reminder of Kontsevich's graph complex GC

5.1 The operad Gra and its action on $T_{\rm poly}(P)$

We start by recalling the graded operad Gra [21, Section 7], [47].

For this purpose, we introduce an auxiliary set gra_n. An element of gra_n is a labelled graph Γ with n vertices and with the additional piece of data: the set of edges of Γ is equipped with a total order. An example of an element in gra_4 is shown on figure 5.1. We will often use Roman numerals to specify total orders on sets of edges. Thus the Roman numerals on figure 5.1 indicate that we chose the total order $(1,1) < (1,2) < (1,3)$.

The space $\mathsf{Gra}(n)$ (for $n \geq 1$) is spanned by elements of gra_n, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$, where the elements Γ^σ and Γ correspond to the same labelled graph but differ only by permutation σ of edges. We also declare that the degree of a graph Γ in $\mathsf{Gra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in Γ. For example, the graph Γ on figure 5.1 has 3 edges. Thus its degree is -3.

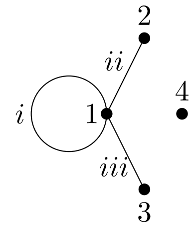

Figure 5.1: The Roman numerals indicate that we chose the total order on the set of edges $(1,1) < (1,2) < (1,3)$

Finally, we set

$$\mathsf{Gra}(0) = \mathbf{0}\,. \tag{5.1}$$

The symmetric group S_n acts on $\mathsf{Gra}(n)$ in the obvious way by rearranging labels on vertices and elementary operadic insertions

$$\circ_i : \mathsf{Gra}(n) \otimes \mathsf{Gra}(k) \to \mathsf{Gra}(n+k-1)$$

are defined using natural operations with labeled graphs (we refer the reader for more details to [21, Section 7]).

To define an action of Gra on $T_{\mathrm{poly}}(P)$ (with $P = \mathbb{K}[[t^1, t^2, \dots, t^d]]$) we identify $T_{\mathrm{poly}}(P)$ with the graded commutative algebra

$$P[\xi_1, \xi_2, \dots, \xi_d]\,, \tag{5.2}$$

where ξ_a's are degree 1 auxiliary variables.

Next, given an element $\Gamma \in \mathsf{gra}_n$ and polyvectors $v_1, \dots, v_n \in P[\xi_1, \xi_2, \dots, \xi_d]$, we set

$$\Gamma(v_1, \dots, v_n) := \mathrm{mult}_n \Big(\Big[\prod_{(i,j) \in E(\Gamma)} \underline{\Delta}_{(i,j)} \Big] \, (v_1 \otimes v_2 \otimes \dots \otimes v_n) \Big)\,, \tag{5.3}$$

where mult_n is the multiplication map

$$\mathrm{mult}_n : \big(T_{\mathrm{poly}}(P)\big)^{\otimes\, n} \to T_{\mathrm{poly}}(P)\,,$$

$E(\Gamma)$ is the set of edges of Γ,

$$\underline{\Delta}_{(i,j)} = \sum_{a=1}^{d} 1 \otimes \dots \otimes 1 \otimes \underbrace{\partial_{\xi_a}}_{i\text{-th slot}} \otimes 1 \otimes \dots \otimes 1 \otimes \underbrace{\partial_{x^a}}_{j\text{-th slot}} \otimes 1 \otimes \dots \otimes 1 + \tag{5.4}$$

$$\sum_{a=1}^{d} 1 \otimes \dots \otimes 1 \otimes \underbrace{\partial_{x^a}}_{i\text{-th slot}} \otimes 1 \otimes \dots \otimes 1 \otimes \underbrace{\partial_{\xi_a}}_{j\text{-th slot}} \otimes 1 \otimes \dots \otimes 1$$

and the order of operators $\underline{\Delta}_{(i,j)}$ coincides with the total order on the set of edges of Γ.

It is not hard to see that equation (5.3) defines an action of the operad Gra on $T_{\mathrm{poly}}(P)$.

Let us also recall that the operad Gra receives a natural embedding

$$\iota : \mathsf{Ger} \to \mathsf{Gra} \tag{5.5}$$

from the operad Ger. Namely, the embedding ι of Ger into Gra is defined on generators by the formulas:

$$\iota(a_1 a_2) := \Gamma_{\bullet\bullet}\,, \qquad \iota(\{a_1, a_2\}) := \Gamma_{\bullet\!-\!\bullet}\,, \tag{5.6}$$

where

$$\Gamma_{\bullet\bullet} = \overset{1}{\bullet}\quad\overset{2}{\bullet} \qquad\qquad \Gamma_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet} \tag{5.7}$$

The Gerstenhaber algebra structure on $T_{\mathrm{poly}}(P)$ is induced by the embedding (5.5).

Remark 5.1 It is easy to see that equation (5.3) defines an action of the operad Gra on polyvector fields on an affine space. Although equation (5.3) does not make sense for an arbitrary (smooth) algebraic variety, the action of $\Gamma_{\bullet\bullet}$ and $\Gamma_{\bullet\!-\!\bullet}$ (5.7) are well defined on polyvector fields of an arbitrary (smooth) algebraic variety.

5.2 The full graph complex fGC

Let us recall that the convolution Lie algebra

$$\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra}) = \prod_{n \ge 1} \mathbf{s}^{2n-2}\Big(\mathsf{Gra}(n)\Big)^{S_n} \tag{5.8}$$

carries the canonical Maurer-Cartan element $\Gamma_{\bullet\!-\!\bullet}$. In other words, $\Gamma_{\bullet\!-\!\bullet}$ has degree 1 in (5.8) and

$$[\Gamma_{\bullet\!-\!\bullet}, \Gamma_{\bullet\!-\!\bullet}] = 0\,. \tag{5.9}$$

So the *full graph complex* fGC is the cochain complex $\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra})$ with the differential

$$\partial = [\Gamma_{\bullet\!-\!\bullet}, \ ]\,. \tag{5.10}$$

According to [47], the cohomology of fGC can be expressed in terms of the cohomology of the subcomplex

$$\mathsf{GC} \subset \mathsf{fGC} \tag{5.11}$$

which consists of infinite sums

$$\sum_{n \ge 4} X_n\,, \qquad X_n \in \mathbf{s}^{2n-2}\Big(\mathsf{Gra}(n)\Big)^{S_n} \tag{5.12}$$

where each graph Γ in the linear combination X_n satisfies these properties

- Γ is connected,
- Γ is 1-vertex irreducible[12], and
- each vertex of Γ has valency ≥ 3.

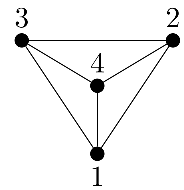

Figure 5.2: We may choose this order on the set of edges: $(1,2) < (1,3) < (1,4) < (2,3) < (2,4) < (3,4)$

The graph complex GC was introduced in [40] by M. Kontsevich.

Example 5.2 Consider the tetrahedron in $\mathsf{Gra}(4)$ depicted on figure 5.2. This vector is invariant with respect to the action of S_4 and hence it can be viewed as vector in fGC. It is not hard to see that this is a degree zero non-trivial cocycle in fGC. Moreover the tetrahedron is connected, 1-vertex irreducible and trivalent. Thus this is an example of a non-trivial degree zero cocycle in GC.

Due to [47, Theorem 1] and [47, Proposition 15] we have

Theorem 5.3 ([47]) *Let $\mathfrak{grt}$ be the Grothendieck-Teichmüller Lie algebra*[13] *[1, Section 4.2] and let σ_n (n is odd ≥ 3) be the Deligne-Drinfeld elements [1, Section 4.2, eq. (15)] of $\mathfrak{grt}$. Then, for Kontsevich's graph complex GC we have*

$$H^0(\mathsf{GC}) \cong \mathfrak{grt}\,, \tag{5.13}$$

Furthermore, if $\widetilde{\sigma}_n$ is the class in $H^0(\mathsf{GC})$ corresponding to $\sigma_n \in \mathfrak{grt}$, then each representative of $\widetilde{\sigma}_n$ has a non-zero coefficient in front of the graph shown on figure 5.3.

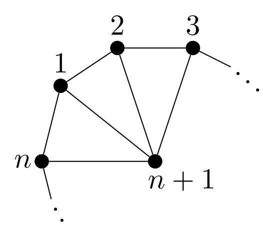

Figure 5.3: Here n is an odd integer ≥ 3. (We do not specify the order on the set of edges.)

[12] A connected graph Γ is called *1-vertex irreducible* if the complement of each vertex of Γ is connected.

[13] Strictly speaking, $\mathfrak{grt}$ is the pro-nilpotent part of the Lie algebra introduced by V. Drinfeld. In [1] and [25], the Lie algebra we use here is denoted by $\mathfrak{grt}_1$.

5.3 The action of fGC and GC on $T_{\mathrm{poly}}(P)$

Let us view $T_{\mathrm{poly}}(P)$ as a $\Lambda\mathsf{Lie}$-algebra. Then, according to Appendix A, the deformation complex of $T_{\mathrm{poly}}(P)$ is

$$\mathbf{Def}_{\Lambda\mathsf{Lie}}(T_{\mathrm{poly}}(P)) = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathrm{End}_{T_{\mathrm{poly}}(P)})\,, \tag{5.14}$$

where $\mathrm{End}_{T_{\mathrm{poly}}(P)}$ is the endomorphism operad of $\mathrm{End}_{T_{\mathrm{poly}}(P)}$ and the differential is given by the adjoint action of the Maurer-Cartan element which corresponds to the composition

$$\mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \to \Lambda\mathsf{Lie} \to \mathrm{End}_{T_{\mathrm{poly}}(P)}\,.$$

The canonical operad morphism defined by (5.3)

$$\mathfrak{a} : \mathsf{Gra} \to \mathrm{End}_{T_{\mathrm{poly}}(P)} \tag{5.15}$$

induces a morphism of graded Lie algebras:

$$\mathfrak{a}_* : \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra}) \to \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathrm{End}_{T_{\mathrm{poly}}(P)})\,. \tag{5.16}$$

Furthermore, since for the generator $\{a_1, a_2\} \in \Lambda\mathsf{Lie}(2)$, $\iota(\{a_1, a_2\}) = \Gamma_{\bullet\!-\!\bullet}$, the map $\mathfrak{a}_*$ sends the Maurer-Cartan element of $\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra})$ to the Maurer-Cartan element of $\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathrm{End}_{T_{\mathrm{poly}}(P)})$. Hence $\mathfrak{a}_*$ is also a map of dg Lie algebras, and restricting $\mathfrak{a}_*$ to

$$\mathsf{GC} \subset \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra}),$$

we get a map (of dg Lie algebras) which we denote by the same letter $\mathfrak{a}_*$

$$\mathfrak{a}_* : \mathsf{GC} \to \mathbf{Def}_{\Lambda\mathsf{Lie}}(T_{\mathrm{poly}}(P))\,. \tag{5.17}$$

5.4 Dg Lie algebras related to fGC

This section is devoted to the auxiliary dg Lie algebras $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ and $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$ which are used in proving a remarkable property of the map from Kontsevich's graph complex GC to the deformation complex of the sheaf of polyvector fields.

First, we recall that the cooperad $\mathsf{Ger}^\vee$ is obtained by taking the linear dual of the operad $\Lambda^{-2}\mathsf{Ger}$. Hence, as graded vector spaces,

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) \cong \prod_{n\ge 1} \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}\,, \tag{5.18}$$

and

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra}) \cong \prod_{n\ge 1} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}\,. \tag{5.19}$$

Next, we identify $\mathsf{Ger}(n)$ with the subspace of the free Gerstenhaber algebra $\mathsf{Ger}(a_1, \dots, a_n)$ spanned by Ger-monomials in which each dummy variable from the set $\{a_1, a_2, \dots, a_n\}$ appears exactly once. We also identify $\Lambda^{-2}\mathsf{Ger}(n)$ with the subspace of the free $\Lambda^{-2}\mathsf{Ger}$-algebra

$\Lambda^{-2}\mathsf{Ger}(b_1, \dots, b_n)$ spanned by $\Lambda^{-2}\mathsf{Ger}$-monomials in which each dummy variable from the set $\{b_1, b_2, \dots, b_n\}$ appears exactly once.

Then vectors in (5.18) are infinite sums

$$\sum_{n \ge 1} Z_n \,, \tag{5.20}$$

$$Z_n = \sum_j Z_{n,j} \otimes w_{n,j} \in \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \,, \tag{5.21}$$

where $Z_{n,j} \in \mathsf{Ger}(n)$, $w_{n,j}$ is a $\Lambda^{-2}\mathsf{Ger}$-monomial of the form

$$\varphi_1(b_{i_{11}}, \dots, b_{i_{1k_1}}) \varphi_2(b_{i_{21}}, \dots, b_{i_{2k_2}}) \dots \varphi_q(b_{i_{q1}}, \dots, b_{i_{qk_q}}) \,, \tag{5.22}$$

$\varphi_1, \dots, \varphi_q$ are $\Lambda^{-1}\mathsf{Lie}$-monomials, and each dummy variable from the set $\{b_1, b_2, \dots, b_n\}$ appears in (5.22) exactly once.

Similarly, vectors in (5.19) are infinite sums

$$\sum_{n \ge 1} Y_n \,, \tag{5.23}$$

$$Y_n = \sum_j Y_{n,j} \otimes w_{n,j} \in \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \,, \tag{5.24}$$

where $Y_{n,j} \in \mathsf{Gra}(n)$ and $w_{n,j}$'s are as above.

The canonical operad morphism

$$\mathrm{Cobar}(\mathsf{Ger}^\vee) \to \mathsf{Ger} \tag{5.25}$$

corresponds to the Maurer-Cartan element

$$\alpha_{\mathsf{Ger}} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\} \,, \tag{5.26}$$

which allows us to equip the graded Lie algebra (5.18) with the differential

$$[\alpha_{\mathsf{Ger}}, \ ] \,. \tag{5.27}$$

We recall [21, Section 11] that the cochain complex (5.18) with the differential (5.27) is called the *extended deformation complex* of the operad Ger.

Using the map

$$\iota_* : \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) \to \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$$

induced by ι (5.5) we get the following Maurer-Cartan element

$$\alpha := \iota_*(\alpha_{\mathsf{Ger}}) = \Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2 + \Gamma_{\bullet\,\bullet} \otimes \{b_1, b_2\} \tag{5.28}$$

in the graded Lie algebra $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$.

Furthermore, just as for $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$, we use α to equip the graded Lie algebra $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$ with the differential:

$$[\alpha, \ ] \,. \tag{5.29}$$

Let us observe that, using the map (5.5), we can embed the vector spaces $\mathsf{Ger}(n) \otimes \mathsf{Ger}^\vee(n)$ and $\mathsf{Gra}(n) \otimes \mathsf{Ger}^\vee(n)$ in the vector space

$$\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Gra}(n)\,. \tag{5.30}$$

Furthermore, it is convenient to represent vectors in (5.30) by formal linear combinations of labeled graphs with two types of edges: solid edges for left tensor factors and dashed edges for right tensor factors.

Using this interpretation, we introduce the following subspaces of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ and $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$, respectively:

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}} \subset \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})\,, \tag{5.31}$$

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}} \subset \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})\,. \tag{5.32}$$

Here $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ (resp. $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$) consists of vectors (5.20) (resp. (5.23)) for which images of Z_n (resp. Y_n) in (5.30) are sums of connected graphs. For example, the vectors α_{Ger} and α belong to $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ and $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$, respectively, while the vector $a_1 a_2 \otimes b_1 b_2 \in \big(\mathsf{Ger}(2) \otimes \Lambda^{-2}\mathsf{Ger}(2)\big)^{S_2}$ does not belong to $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$.

It is easy to see that $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ (resp. $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$) is a subcomplex of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ (resp. $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$).

For our purposes, we also need the subspace

$$\Xi_{\mathrm{conn}} \subset \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}\,. \tag{5.33}$$

This subspace consists of sums (5.20) in $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ for which each $\Lambda^{-1}\mathsf{Lie}$-monomial in (5.22) has length ≥ 2.

Even though $\alpha_{\mathsf{Ger}} \notin \Xi_{\mathrm{conn}}$, the subspace Ξ_{conn} is a subcomplex of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$. This subcomplex plays an important role in establishing a link between the cohomology of the graph complex fGC and the cohomology of the extended deformation complex of Ger.

Next, we consider the subspace of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$ which consists of sums (5.23) in $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$ satisfying the following property:

Property 5.4 *If the variable b_i forms a $\Lambda^{-1}\mathsf{Lie}$-word of length 1 in $w_{n,j}$, then, in each graph of $Y_{n,j}$, the vertex with label i is at least trivalent.*

It is not hard to see that this subspace is subcomplex of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$ and we denote it by

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \tag{5.34}$$

We introduce the map $\mathfrak{R}$ of cochain complexes:

$$\mathfrak{R} : \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \to \mathsf{fGC}$$
$$\mathfrak{R}(f) = f\Big|_{\Lambda^2\mathsf{coCom}} \tag{5.35}$$

and observe that the subcomplex

$$\ker(\mathfrak{R}) \subset \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \tag{5.36}$$

receives an obvious map ψ from the cochain complex Ξ_{conn} (5.33) which is induced by the map ι (5.5).

We claim that

Proposition 5.5 *The map*

$$\psi : \Xi_{\mathrm{conn}} \to \ker(\mathfrak{R}) \tag{5.37}$$

induces an isomorphism on the level of cohomology.

Proof. This statement is an analogue of Proposition 13.1 in [21].

To prove this statement we only need the following minor modification of the proof of [21, Proposition 13.1]: we need to replace in *loc. cit.* the dg operad $\mathsf{fgraphs}$ by the dg operad graphs (see also [21, Section 9.4]). Thus, since the embedding $\mathsf{graphs} \hookrightarrow \mathsf{fgraphs}$ is a quasi-isomorphism, the modified proof goes through. $\square$

6 The chain map $\Theta : \mathsf{GC} \to \mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})$

Let us recall that, due to Theorem 3.5, the sheaf of dg Gerstenhaber algebras

$$\mathcal{FR} := \Big(\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T_{\mathrm{poly}}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]} \tag{6.1}$$

is a resolution of the sheaf of Gerstenhaber algebras $\mathcal{T}_{\mathrm{poly}}$ on a smooth algebraic variety X.

In this section, we view $\mathcal{FR}$ as the sheaf of $\Lambda\mathsf{Lie}$-algebras and construct a map of dg Lie algebras

$$\Theta : \mathsf{GC} \to \mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})\,. \tag{6.2}$$

For this purpose, we denote by $\mathcal{FR}'$ the following sheaf of dg $\Lambda\mathsf{Lie}$ algebras

$$\mathcal{FR}' = \Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}}) \hat{\otimes} T_{\mathrm{poly}}(P) \tag{6.3}$$

with the de Rham differential d.

We recall that the sheaf of dg $\Lambda\mathsf{Lie}$ algebras $\mathcal{FR}$ (6.1) is obtained from $\mathcal{FR}'$ in two steps. First, we twist[14] the dg $\Lambda\mathsf{Lie}$-algebra structure by the Maurer-Cartan element ω introduced in Theorem 2.12. Under this procedure, the $\Lambda\mathsf{Lie}$-bracket remains the same $\{\ ,\ \}_{SN}$ and the differential d gets replaced by

$$d + \{\omega,\ \}_{SN}\,. \tag{6.4}$$

Second, we apply trimming (see Section 1.2) to the resulting sheaf of dg $\Lambda\mathsf{Lie}$-algebras with respect to the set of degree -1 derivations $i_{\bar{\mathfrak{v}}}$, $\mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K})$ and get (6.1).

So to construct (6.2), we define an auxiliary map of dg Lie algebras

$$\Theta' : \mathsf{GC} \to \mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)\,. \tag{6.5}$$

via extending the map $\mathfrak{a}_*$ (5.17) by linearity over $\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})$. Here the codomain

$$\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)$$

carries the differential

$$[d + Q,\ ]\,, \tag{6.6}$$

[14]See Appendix C for terminology and details of the twisting procedure.

with d being the de Rham differential and Q being the coderivation of $\Lambda^2\mathsf{coCom}(\mathcal{FR}')$ coming from the $\Lambda\mathsf{Lie}$-bracket $\{\ ,\ \}_{SN}$.

Since Θ' is obtained via extending $\mathfrak{a}_*$ (5.17) by linearity over $\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})$ and $\mathfrak{a}_*$ is a chain map, we conclude that Θ' intertwines the differential ∂ (5.10) with the differential $d+Q$, i.e.

$$[d+Q, \Theta'(\gamma)] = \Theta'(\partial\gamma)\,, \qquad \forall\ \ \gamma \in \mathsf{GC}\,. \tag{6.7}$$

For the map Θ', we have the following statement:

Proposition 6.1 *Let $\omega = \omega^a \partial_{t^a}$ be the global section of the sheaf*

$$\Omega^1(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes} T^{1,0}(P)$$

introduced in Theorem 2.12. Then the formula

$$(\Theta')^{\omega}(\gamma) = e^{-\mathbf{s}^{-2}\,\omega}\,\Theta'(\gamma)\,e^{\mathbf{s}^{-2}\,\omega}\,, \qquad \gamma \in \mathsf{GC} \tag{6.8}$$

defines a map of dg Lie algebras

$$(\Theta')^{\omega} : \mathsf{GC} \to \mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)\,, \tag{6.9}$$

where $\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)$ is considered with the differential

$$[d + \{\omega,\ \}_{SN} + Q,\ ]\,.$$

Furthermore, $(\Theta')^{\omega}$ descends to a map of dg Lie algebras

$$\mathsf{GC} \ \to\ \mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR})\big)\,, \tag{6.10}$$

where

$$\mathcal{FR} = \Big(\Omega^\bullet(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes} T_{\mathrm{poly}}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]}\,.$$

Proof. We remark that, using the degree of exterior forms, we equip the sheaf $\mathcal{FR}'$ of dg $\Lambda\mathsf{Lie}$-algebras with the natural decreasing filtration. This filtration is complete, and hence, we may apply to $\mathcal{FR}'$ the operation of twisting (see Appendix C).

Let us denote by p the canonical projection

$$p : \Lambda^2\mathsf{coCom}(\mathcal{FR}') \to \mathcal{FR}'\,, \tag{6.11}$$

and prove that for all $n \ge 1$

$$p \circ \Theta'(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,\omega)^n\big) = 0\,. \tag{6.12}$$

For this purpose, we recall that an action of a graph Γ on a collection of polyvector fields is expressed in terms of the operators (5.4).

So we will keep track of terms involving the sum

$$\sum_{a=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\xi_a}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{x^a}}_{j\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1$$

by choosing a direction on the edge (i,j) from vertex i to vertex j. Similarly, we will keep track of terms involving the sum

$$\sum_{a=1}^{d} 1\otimes\dots\otimes 1\otimes \underbrace{\partial_{x^a}}_{i\text{-th slot}} \otimes 1\otimes\dots\otimes 1\otimes \underbrace{\partial_{\xi_a}}_{j\text{-th slot}} \otimes 1\otimes\dots\otimes 1$$

by choosing a direction on the edge (i,j) from vertex j to vertex i.

Since ω is vector-valued, equation (6.12) is a consequence of the following simple combinatorial fact:

Claim 6.2 *If each vertex of a graph Γ has valency ≥ 3 then edges of Γ cannot be oriented in such a way that all vertices have exactly one outgoing edge.* $\square$

Thus (6.12) indeed holds.

Now, it is easy to see that equations (6.7), (6.12), and Corollary C.2 from Appendix C imply that formula (6.8) indeed defines a map from the graph complex GC to the cochain complex $\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)$ with the differential

$$[d+\{\omega,\ \}_{SN}+Q,\ ]\,.$$

The compatibility of this map with the Lie brackets is obvious.

It remains to prove that for every cochain $\gamma\in\mathsf{GC}$ and for any set $v_1,\dots,v_n$ of local sections of $\mathcal{FR}$ (6.1) the section

$$v = p\circ(\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big) \tag{6.13}$$

satisfies the conditions

$$i_{\overline{\mathfrak{v}}}(v)=0 \tag{6.14}$$

and

$$i_{\overline{\mathfrak{v}}}(dv+\{\omega,v\}_{SN})=0 \qquad \forall\ \ \mathfrak{v}\in\mathfrak{gl}_d(\mathbb{K}). \tag{6.15}$$

Let

$$\gamma=\sum_{\tau\in S_N}\tau(\Gamma)$$

for an element $\Gamma\in\mathsf{gra}_N$. Then

$$v=\frac{1}{r!}\,p\circ\Theta'(\gamma)\big(\mathbf{s}^2((\mathbf{s}^{-2}\omega)^r\,\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)=$$

$$\frac{1}{r!}\,\gamma(\underbrace{\omega,\dots,\omega}_{r\text{ times}},v_1,\dots,v_n)$$

where $r=N-n$ and the action of γ on local sections of $\mathcal{FR}$ is obtained via extending (5.3) by linearity over $\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})$.

Since for all $\mathfrak{v}\in\mathfrak{gl}_d(\mathbb{K})$ we have $i_{\overline{\mathfrak{v}}}v_j=0$, therefore

$$i_{\overline{\mathfrak{v}}}\big(\gamma(\underbrace{\omega,\dots,\omega}_{r\text{ times}},v_1,\dots,v_n)\big)=\sum_{k=0}^{r-1}\gamma(\underbrace{\omega,\dots,\omega}_{k\text{ times}},i_{\overline{\mathfrak{v}}}\omega,\underbrace{\omega,\dots,\omega}_{r-k-1\text{ times}},v_1,\dots,v_n)\,.$$

On the other hand, by Corollary 2.13,

$$i_{\overline{\mathfrak{v}}}\,\omega = -\mathfrak{v}^a_b t^b \frac{\partial}{\partial t^a}\,.$$

Hence equation (6.14) holds simply because all vertices of Γ have valency ≥ 3.

To prove (6.15), we recall that $(\Theta')^{\omega}$ is a chain map from the graph complex GC to the cochain complex $\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR}')\big)$ with the differential

$$[d + \{\omega,\ \}_{SN} + Q,\ ]\,.$$

Therefore[15]

$$\begin{aligned}
&(d + \{\omega,\ \}_{SN}) \circ p \circ (\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\\
&= p \circ \Big((d + \{\omega,\ \}_{SN}) \circ (\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\Big)\\
&=\ p \circ \Big((d + \{\omega,\ \}_{SN} + Q) \circ (\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\Big)\\
&\qquad - (-1)^{|\gamma|} p \circ \Big((\Theta')^{\omega}(\gamma) \circ (d + \{\omega,\ \}_{SN} + Q)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\Big)\\
&\qquad + (-1)^{|\gamma|} p \circ \Big((\Theta')^{\omega}(\gamma) \circ (d + \{\omega,\ \}_{SN} + Q)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\Big)\\
&\qquad - p \circ \Big(Q \circ (\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\Big)\\
&=\ p \circ (\Theta')^{\omega}(\partial\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\\
&\qquad + \sum_{k=1}^{n} \pm \frac{1}{r!}\gamma(\underbrace{\omega,\dots,\omega}_{r \text{ times}}, v_1, \dots, v_{k-1}, dv_k + \{\omega, v_k\}_{SN}, v_{k+1}, \dots, v_n)\\
&\qquad + \sum_{1\le j<k\le n} \pm \frac{1}{(r+1)!}\gamma(\underbrace{\omega,\dots,\omega}_{(r+1) \text{ times}}, \{v_j, v_k\}_{SN}, v_1 \dots, v_{j-1}, v_{j+1}, \dots, v_{k-1}, v_{k+1}, \dots, v_n)\\
&\qquad + \sum_{k=1}^{n} (-1)^{\varepsilon_k} \pm \frac{1}{(r+1)!}\{\gamma(\underbrace{\omega,\dots,\omega}_{(r+1) \text{ times}}, v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n), v_k\}_{SN}\,.
\end{aligned} \tag{6.16}$$

[15] In computation (6.16), we put $\pm$ in front of terms for which sign factors do not play an important role.

Hence we get

$$
\begin{aligned}
&(d+\{\omega,\ \}_{SN})\circ p\circ(\Theta')^{\omega}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,v_1\,\mathbf{s}^{-2}\,v_2\,\dots\,\mathbf{s}^{-2}\,v_n)\big)\\
&= \frac{1}{r!}(\partial\gamma)(\underbrace{\omega,\dots,\omega}_{r\text{ times}},v_1,\dots,v_n)\\
&\quad+\sum_{k=1}^{n}\pm\frac{1}{r!}\gamma(\underbrace{\omega,\dots,\omega}_{r\text{ times}},v_1,\dots,v_{k-1},dv_k+\{\omega,v_k\}_{SN},v_{k+1},\dots,v_n)\\
&\quad+\sum_{1\le j<k\le n}\pm\frac{1}{(r+1)!}\gamma(\underbrace{\omega,\dots,\omega}_{(r+1)\text{ times}},\{v_j,v_k\}_{SN},v_1\dots,v_{j-1},v_{j+1},\dots,v_{k-1},v_{k+1},\dots,v_n)\\
&\quad+\sum_{k=1}^{n}(-1)^{\varepsilon_k}\pm\frac{1}{(r+1)!}\{\gamma(\underbrace{\omega,\dots,\omega}_{(r+1)\text{ times}},v_1,\dots,v_{k-1},v_{k+1},\dots,v_n),v_k\}_{SN}\,.
\end{aligned}
\tag{6.17}
$$

Thus we see that (6.15) follows from equations (6.14),

$$
i_{\overline{\mathfrak{v}}}\big(dv_k+\{\omega,v_k\}_{SN}\big)=0
$$

and the fact that $i_{\overline{\mathfrak{v}}}$ is a derivation of the bracket $\{\ ,\ \}_{SN}$.

Proposition 6.1 is proved. □

Composing the map of dg Lie algebras (6.10) with the canonical morphism

$$
\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{FR})\big)\to\mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})
$$

we get the desired map of dg Lie algebras

$$
\Theta:\mathsf{GC}\to\mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})\,. \tag{6.18}
$$

6.1 Extending Θ to $\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra})_{\ge 3}$

Although $\mathcal{FR}$ is also a sheaf of Gerstenhaber algebras, there is no natural way of extending the map Θ (6.18) to a map

$$
\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra})\to\mathbf{Def}_{\mathsf{Ger}}(\mathcal{FR}) \tag{6.19}
$$

from the dg Lie algebra $\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra})$ to the deformation complex $\mathbf{Def}_{\mathsf{Ger}}(\mathcal{FR})$ of $\mathcal{FR}$ (viewed as the sheaf of Gerstenhaber algebras).

However, it is possible to extend Θ (6.18) to a map from a dg Lie subalgebra $\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra})_{\ge 3}$ (5.34) to $\mathbf{Def}_{\mathsf{Ger}}(\mathcal{FR})$.

To construct this map, we extend $\mathfrak{a}$ (5.15) by linearity over $\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})$ to

$$
\mathsf{Gra}(n)\to\mathrm{Hom}\big((\mathcal{FR}')^{\otimes n},\mathcal{FR}'\big)\,, \tag{6.20}
$$

where $\mathcal{FR}'$ is the auxiliary sheaf of dg Gerstenhaber algebras defined in (6.3).

Next, using (6.20), we get an auxiliary map of dg Lie algebras

$$
\Theta'_{\mathsf{Ger}}:\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra})\to\mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR}')\big)\,, \tag{6.21}
$$

where the codomain

$$\mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR}')\big)$$

carries the differential

$$[d + Q_{\mathsf{Ger}},\ ]\,, \tag{6.22}$$

with d being the de Rham differential and Q_{Ger} being the coderivation of $\mathsf{Ger}^{\vee}(\mathcal{FR}')$ coming from the Gerstenhaber algebra structure on $\mathcal{FR}'$.

We now recall that the sheaf of Gerstenhaber algebras $\mathcal{FR}$ is obtained from $\mathcal{FR}'$ (6.3) in two steps. First, we need to twist $\mathcal{FR}'$ by the Maurer-Cartan element ω defined in Theorem 2.12. Second, we apply trimming with respect to the derivations coming from the action of $\mathfrak{gl}_d(\mathbb{K})$.

We have the following analog of Proposition 6.1:

Proposition 6.3 *Let $\omega = \omega^a \partial_{t^a}$ be the global section of the sheaf*

$$\Omega^1(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes} T^{1,0}(P)$$

introduced in Theorem 2.12 and $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3}$ *be the dg Lie subalgebra of* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ *introduced in* (5.34). *Then the formula*

$$(\Theta'_{\mathsf{Ger}})^{\omega}(\gamma) = e^{-\mathbf{s}^{-2}\,\omega}\,\Theta'_{\mathsf{Ger}}(\gamma)\, e^{\mathbf{s}^{-2}\,\omega}\,, \qquad \gamma \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3} \tag{6.23}$$

defines a map of dg Lie algebras

$$(\Theta')^{\omega}_{\mathsf{Ger}} : \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3} \to \mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR}')\big)\,, \tag{6.24}$$

where $\mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR}')\big)$ *is considered with the differential*

$$[d + \{\omega,\ \}_{SN} + Q_{\mathsf{Ger}},\ ]\,. \tag{6.25}$$

Furthermore, $(\Theta')^{\omega}_{\mathsf{Ger}}$ *descends to a map of dg Lie algebras*

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3} \ \to\ \mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR})\big)\,, \tag{6.26}$$

where

$$\mathcal{FR} = \Big(\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes} T_{\mathrm{poly}}(P)\Big)^{[\mathfrak{gl}_d(\mathbb{K})]}\,.$$

Proof. Just as in the proof of Proposition 6.1, Claim 6.2 implies that for every vector $\gamma \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3}$, we have

$$p \circ \Theta'_{\mathsf{Ger}}(\gamma)\big(\mathbf{s}^2(\mathbf{s}^{-2}\,\omega)^n\big) = 0\,, \tag{6.27}$$

where p is the canonical projection

$$p : \mathsf{Ger}^{\vee}(\mathcal{FR}') \to \mathcal{FR}'\,, \tag{6.28}$$

and $\mathbf{s}^2(\mathbf{s}^{-2}\,\omega)^n$ is a global section of $\Lambda^2\mathsf{coCom}(\mathcal{FR}') \subset \mathsf{Ger}^{\vee}(\mathcal{FR}')$.

Hence Theorem C.3 from Appendix C.2 implies that the assignment

$$\gamma \ \mapsto \ e^{-\mathbf{s}^{-2}\,\omega}\,\Theta'_{\mathsf{Ger}}(\gamma)\,e^{\mathbf{s}^{-2}\,\omega} \tag{6.29}$$

is a map of dg Lie algebras from $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3}$ to $\mathsf{coDer}\big(\mathsf{Ger}^{\vee}(\mathcal{FR}')\big)$, where the codomain is considered with the differential (6.25).

It remains to prove that for every cochain $\gamma \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\ge 3}$ and any local section $W \in \mathsf{Ger}^{\vee}(\mathcal{FR})$ the section of $\mathcal{FR}'$

$$v = p \circ (\Theta'_{\mathsf{Ger}})^{\omega}(\gamma)\big(W\big) \tag{6.30}$$

satisfies the conditions

$$i_{\overline{\mathfrak{v}}}(v) = 0 \tag{6.31}$$

and

$$i_{\overline{\mathfrak{v}}}(dv + \{\omega, v\}_{SN}) = 0 \qquad \forall \ \ \mathfrak{v} \in \mathfrak{gl}_d(\mathbb{K}). \tag{6.32}$$

The latter can be shown by going through the corresponding steps *mutatis mutandis* in the proof of Proposition 6.1. □

7 For every cocycle $\gamma \in \mathsf{GC}$ the cocycle $\Theta(\gamma)$ induces a derivation of the Gerstenhaber algebra $H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}})$

Let us recall (see Appendix B.5) that for every degree k cocycle $\gamma \in \mathsf{GC}$ the cocycle $\Theta(\gamma) \in \mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})$ induces a degree k derivation of the $\Lambda\mathsf{Lie}$-algebra

$$H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}})\,. \tag{7.1}$$

We will denote this derivation by D_{γ} .

More precisely, if v is a cocycle in

$$\check{C}^{\bullet}(X, \mathcal{FR}) \tag{7.2}$$

representing a class $[v]$ in (7.1) then $\mathsf{D}_{\gamma}([v])$ is represented by

$$\sum_{n \ge 1} \frac{1}{n!}\gamma(\underbrace{\omega, \dots, \omega}_{n \text{ times}}, v\,) \tag{7.3}$$

where the action of γ on local sections of $\mathcal{FR}$ is obtained via extending (5.3) by linearity over $\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})$ and ω is the global section of the sheaf $\Omega^1(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{1,0}(P)$ introduced in Theorem 2.12.

Our goal is to prove that

Theorem 7.1 *For every cocycle $\gamma \in \mathsf{GC}$, the map*

$$\mathsf{D}_{\gamma} : H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}}) \to H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}}) \tag{7.4}$$

defined by (7.3) *is a derivation of the Gerstenhaber algebra structure on* $H^{\bullet}(X, \mathcal{T}_{\mathrm{poly}})$

A proof of this theorem is given in Section 7.1 below. It is based on a technical claim which we present now.

First, we recall that the dg Lie algebras fGC, GC, $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$, and $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$ carry a natural decreasing filtration "by arity":

$$\mathcal{F}_m \mathsf{fGC} := \prod_{n \ge m+1} \mathbf{s}^{2n-2} \Big(\mathsf{Gra}(n) \Big)^{S_n}, \qquad \mathcal{F}_m \mathsf{GC} = \mathsf{GC} \, \cap \, \mathcal{F}_m \mathsf{fGC} \,, \tag{7.5}$$

$$\mathcal{F}_m \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) := \prod_{n \ge m+1} \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2} \mathsf{Ger}(n) \Big)^{S_n}, \tag{7.6}$$

and

$$\mathcal{F}_m \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra}) := \prod_{n \ge m+1} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2} \mathsf{Ger}(n) \Big)^{S_n}. \tag{7.7}$$

Second, we observe that every cochain $\gamma \in \mathsf{fGC}$ may be extended "by zero" to a cochain $\widetilde{\gamma}$ in (5.19). Indeed, the desired element $\widetilde{\gamma}$ is defined by declaring that it vanishes on all vectors in $\mathsf{Ger}^\vee$ which involve at least one $\Lambda\mathsf{coLie}$-monomial of length ≥ 2, and setting

$$\widetilde{\gamma} \Big|_{\Lambda^2 \mathsf{coCom}} = \gamma \,. \tag{7.8}$$

It is obvious that for every cochain $\gamma \in \mathsf{GC}$ we have

$$\widetilde{\gamma} \, \in \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \,.$$

Finally, we formulate the technical statement which is used in the proof of Theorem 7.1:

Proposition 7.2 *Let*

$$\mathfrak{R} : \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \to \mathsf{fGC}$$

be the map of cochain complexes defined in (5.35) *and* γ *be a degree* q *cocycle in* $\mathcal{F}_m \mathsf{GC}$ *. There exists a degree* q *cochain* $\theta \in \ker(\mathfrak{R})$ *and a degree* $q{+}1$ *cocycle* $x \in \Xi_{\mathrm{conn}} \cap \mathcal{F}_{m+1} \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ *such that*

$$\partial \, \widetilde{\gamma} \, = \psi(x) + \partial \theta \,, \tag{7.9}$$

where the vector $\widetilde{\gamma} \in \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3}$ *is obtained via extending* γ *"by zero" and* ψ *is the embedding* (5.37)*.*

Proof. Since $\gamma \in \mathcal{F}_m \mathsf{GC}$,

$$\widetilde{\gamma} \, \in \mathcal{F}_m \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3} \,. \tag{7.10}$$

Furthermore, γ is a cocycle in GC. Hence

$$\partial \, \widetilde{\gamma} \, \in \ker(\mathfrak{R}) \,. \tag{7.11}$$

Therefore, by Proposition 5.5, there exists a degree q cochain $\theta' \in \ker(\mathfrak{R})$ and a degree $(q+1)$ cocycle $x' \in \Xi_{\mathrm{conn}} \cap \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ such that

$$\partial \, \widetilde{\gamma} \, = \psi(x') + \partial \theta' \,. \tag{7.12}$$

Let us now observe that

$$\partial\Big(\mathcal{F}_m \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})\Big) \subset \mathcal{F}_{m+1}\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})\,. \tag{7.13}$$

Hence, using inclusion (7.10), we deduce that the restriction of $\psi(x')$ to $\mathsf{Ger}^\vee(n)$ for $n \le m+1$ gives us an exact cocycle in

$$\prod_{n=2}^{m+1} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n} \cap \ker(\mathfrak{R}) \tag{7.14}$$

Therefore, applying Proposition 5.5 again, we conclude that there exists a degree q cochain $\theta'' \in \ker(\mathfrak{R})$ and a degree $q+1$ cocycle $x \in \Xi_{\mathrm{conn}} \cap \mathcal{F}_{m+1}\,\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ such that

$$\psi(x) = \psi(x') - \partial(\theta'')\,.$$

Thus setting $\theta = \theta' + \theta''$ we get the desired equation (7.9). $\square$

7.1 Proof of Theorem 7.1

The map (7.4) is a derivation of the $\Lambda\mathsf{Lie}$-bracket on $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$ since D_γ comes from a cocycle in $\mathbf{Def}_{\Lambda\mathsf{Lie}}(\mathcal{FR})$. Thus it remains to prove that D_γ is a derivation for the commutative algebra structure on $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$.

For this purpose we start with two cocycles in the Čech complex

$$v^1, v^2 \in \check{C}^\bullet(X, \mathcal{FR})$$

and consider the class

$$\mathsf{D}_\gamma([v^1] \cup [v^2]) - \mathsf{D}_\gamma([v^1]) \cup [v^2] - (-1)^{|v^1||\gamma|}[v^1] \cup \mathsf{D}_\gamma([v^2]) \;\in\; H^\bullet(X, \mathcal{FR})$$

where $\cup$ is the multiplication on $H^\bullet(X, \mathcal{FR})$ induced by the $\wedge$-product on $\mathcal{FR}$.

This class is represented by the cocycle v given by the equation

$$\begin{aligned} v_{\alpha_0 \dots \alpha_t} := \; & \sum_{0 \le k \le t} \sum_{n \ge 1} \frac{(-1)^{k|v^2|}}{n!} \gamma(\underbrace{\omega, \dots, \omega}_{n \text{ times}}, v^1_{\alpha_0 \dots \alpha_k} v^2_{\alpha_k \dots \alpha_t}) \\ & - \sum_{0 \le k \le t} \sum_{n \ge 1} \frac{(-1)^{k|v^2|}}{n!} \gamma(\underbrace{\omega, \dots, \omega}_{n \text{ times}}, v^1_{\alpha_0 \dots \alpha_k}) v^2_{\alpha_k \dots \alpha_t} \\ & - (-1)^{|\gamma||v^1|} \sum_{0 \le k \le t} \sum_{n \ge 1} \frac{(-1)^{k(|v^2|+|\gamma|)}}{n!} v^1_{\alpha_0 \dots \alpha_k} \gamma(\underbrace{\omega, \dots, \omega}_{n \text{ times}}, v^2_{\alpha_k \dots \alpha_t})\,. \end{aligned} \tag{7.15}$$

So, our goal is to show that the cocycle v is exact.

To prove this claim, we rewrite (7.15) using the fact that the element

$$\widetilde{\gamma} \in \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\ge 3}$$

is obtained via extending $\gamma \in \mathsf{GC}$ "by zero":

$$
\begin{aligned}
v_{\alpha_0 \dots \alpha_t} = & \sum_{0 \le k \le t} (-1)^{k|v^2|} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} (v^1_{\alpha_0 \dots \alpha_k} v^2_{\alpha_k \dots \alpha_t}) \\
& - \sum_{0 \le k \le t} (-1)^{k|v^2|} \big(p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} (v^1_{\alpha_0 \dots \alpha_k}) \big)\, v^2_{\alpha_k \dots \alpha_t} \\
& - (-1)^{|\widetilde{\gamma}||v^1|} \sum_{0 \le k \le t} (-1)^{k(|v^2| + |\widetilde{\gamma}|)} v^1_{\alpha_0 \dots \alpha_k} \big(p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} (v^2_{\alpha_k \dots \alpha_t}) \big) \\
= & \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k - 1)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \circ \mathscr{Q}^{\omega}_{\mathsf{Ger}} \big(\langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \big) \\
& - \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k - 1)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \big(\langle d v^1_{\alpha_0 \dots \alpha_k} + \{\omega, v^1_{\alpha_0 \dots \alpha_k}\}_{SN}, v^2_{\alpha_k \dots \alpha_t} \rangle \big) \\
& - \sum_{0 \le k \le t} (-1)^{k(|v^2| + 1)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \big(\langle v^1_{\alpha_0 \dots \alpha_k}, d v^2_{\alpha_k \dots \alpha_m} + \{\omega, v^2_{\alpha_k \dots \alpha_t}\}_{SN} \rangle \big) \\
& + \sum_{0 \le k \le t} (-1)^{k|v^2| + |v^1| - k + |\widetilde{\gamma}|} p \circ \mathscr{Q}^{\omega}_{\mathsf{Ger}} \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \\
= & - \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k)} p \circ \Big(e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \mathscr{Q}^{\omega}_{\mathsf{Ger}} \\
& - (-1)^{|\widetilde{\gamma}|} \mathscr{Q}^{\omega}_{\mathsf{Ger}} \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \Big) \langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \,,
\end{aligned}
\tag{7.16}
$$

where

$$
\langle s_1, s_2 \rangle := \{b_1, b_2\}^* \otimes s_1 \otimes s_2 \in \mathsf{Ger}^{\vee}(\mathcal{FR})
$$

for two local sections s_1, s_2 of $\mathcal{FR}$ and $\mathscr{Q}^{\omega}_{\mathsf{Ger}}$ is the codifferential of $\mathsf{Ger}^{\vee}(\mathcal{FR})$ given by

$$
\mathscr{Q}^{\omega}_{\mathsf{Ger}} := e^{-\mathbf{s}^{-2}\omega} \circ (d + Q_{\mathsf{Ger}}) \circ e^{\mathbf{s}^{-2}\omega} = d + \{\omega, \ \}_{SN} + Q_{\mathsf{Ger}} \,. \tag{7.17}
$$

Thus we get

$$
v_{\alpha_0 \dots \alpha_t} = (-1)^{|\gamma|} \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\partial \widetilde{\gamma}) e^{\mathbf{s}^{-2}\omega} \big(\langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \big) \,. \tag{7.18}
$$

Hence, due to Proposition 7.2,

$$
\begin{aligned}
v_{\alpha_0 \dots \alpha_t} = & (-1)^{|\gamma|} \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\psi(x)) e^{\mathbf{s}^{-2}\omega} \big(\langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \big) \\
& + (-1)^{|\gamma|} \sum_{0 \le k \le t} (-1)^{k|v^2| + (|v^1| - k)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\partial \theta) e^{\mathbf{s}^{-2}\omega} \big(\langle v^1_{\alpha_0 \dots \alpha_k}, v^2_{\alpha_k \dots \alpha_t} \rangle \big) \,,
\end{aligned}
\tag{7.19}
$$

where $x \in \Xi_{\mathrm{conn}} \cap \mathcal{F}_{m+1} \operatorname{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$ and $\theta \in \ker(\mathfrak{R})$.

Using the inclusion $x \in \Xi_{\text{conn}}$ and the compatibility of Θ'_{Ger} with the differentials we conclude that

$$\begin{aligned}
v_{\alpha_0\ldots\alpha_t} = \; & (-1)^{|\gamma|} \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ \Theta'_{\mathsf{Ger}}(\psi(x))\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big) \\
& + (-1)^{|\gamma|} \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ \mathscr{Q}^{\omega}_{\mathsf{Ger}} \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega}\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big) \\
& + \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ e^{-\mathbf{s}^{-2}\omega} \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega} \circ \hat{\mathscr{Q}}^{\omega}_{\mathsf{Ger}}\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big)
\end{aligned} \tag{7.20}$$

Since $\mathsf{GC} = \mathcal{F}_2\mathsf{GC}$ we have $x \in \Xi_{\text{conn}} \cap \mathcal{F}_3 \operatorname{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ and hence

$$\Theta'_{\mathsf{Ger}}(\psi(x))\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big) = 0\,.$$

In addition, $\theta \in \ker(\mathfrak{R})$. Therefore,

$$\begin{aligned}
v_{\alpha_0\ldots\alpha_t} = \; & (-1)^{|\gamma|} \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} (d + \{\omega,\ \}_{SN})\Big(p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega}\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big)\Big) \\
& - \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega} \circ \big(\langle (dv^1_{\alpha_0\ldots\alpha_k} + \{\omega, v^1_{\alpha_0\ldots\alpha_k}\}_{SN}), v^2_{\alpha_k\ldots\alpha_t}\rangle\big) \\
& - \sum_{0\le k\le t} (-1)^{k|v^2|} p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega} \circ \big(\langle v^1_{\alpha_0\ldots\alpha_k}, (dv^2_{\alpha_k\ldots\alpha_t} + \{\omega, v^2_{\alpha_k\ldots\alpha_t}\}_{SN})\rangle\big)\,.
\end{aligned} \tag{7.21}$$

Since both v^1 and v^2 are cocycles in the Čech complex $\check{C}^\bullet(X, \mathcal{FR})$, we have

$$dv^1_{\alpha_0\ldots\alpha_k} + \{\omega, v^1_{\alpha_0\ldots\alpha_k}\}_{SN} = -(\check{\partial} v^1)_{\alpha_0\ldots\alpha_k}$$

and

$$dv^2_{\alpha_k\ldots\alpha_t} + \{\omega, v^2_{\alpha_k\ldots\alpha_t}\}_{SN} = -(\check{\partial} v^2)_{\alpha_k\ldots\alpha_t}\,.$$

Hence, equation (7.21) implies that

$$\begin{aligned}
v_{\alpha_0\ldots\alpha_t} = \; & (-1)^{|\gamma|} \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} (d + \{\omega,\ \}_{SN})\Big(p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega}\big(\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big)\Big) \\
& + \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega} \circ \big(\langle (\check{\partial} v^1)_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big) \\
& + \sum_{0\le k\le t} (-1)^{k|v^2|} p \circ \Theta'_{\mathsf{Ger}}(\theta) e^{\mathbf{s}^{-2}\omega} \circ \big(\langle v^1_{\alpha_0\ldots\alpha_k}, (\check{\partial} v^2)_{\alpha_k\ldots\alpha_t}\rangle\big) \\
= \; & du_{\alpha_0\ldots\alpha_t} + \{\omega, u_{\alpha_0\ldots\alpha_t}\}_{SN} + (\check{\partial} u)_{\alpha_0\ldots\alpha_t}\,,
\end{aligned} \tag{7.22}$$

where

$$u_{\alpha_0\ldots\alpha_t} := (-1)^{|\gamma|} \sum_{0\le k\le t} (-1)^{k|v^2|+(|v^1|-k)} p \circ \Theta'_{\mathsf{Ger}}(\theta)\big(e^{\mathbf{s}^{-2}\omega}\langle v^1_{\alpha_0\ldots\alpha_k}, v^2_{\alpha_k\ldots\alpha_t}\rangle\big)\,. \tag{7.23}$$

Thus the cocycle v is indeed exact and Theorem 7.1 follows.

□

8 The action of the Deligne-Drinfeld elements

Let $\mathfrak{grt}$ be the Grothendieck-Teichmüller Lie algebra[16] [1, Section 4.2] and let σ_n (n is odd ≥ 3) be the Deligne-Drinfeld elements [1, Section 4.2, eq. (15)] of $\mathfrak{grt}$. Let us denote by γ_n any cocycle which represents the cohomology class $\widetilde{\sigma}_n \in H^0(\mathsf{GC})$ corresponding to $\sigma_n \in \mathfrak{grt}$.

According to Theorem 7.1, $\Theta(\gamma_n)$ induces a degree zero derivation $\mathcal{D}_n$ of the Gerstenhaber algebra

$$H^{\bullet}(X, \mathcal{T}_{\rm poly})\,. \tag{8.1}$$

The following theorem gives a natural geometric interpretation of this derivation:

Theorem 8.1 *Let n be an odd integer ≥ 3. Then the action of $\mathcal{D}_n$ on (8.1) is a non-zero scalar multiple of the contraction with the n-th component of the Chern character.*

Theorem 8.1 has the following obvious corollary:

Corollary 8.2 *Let X be a smooth algebraic variety over $\mathbb{K}$ and n be an odd integer ≥ 3. Then the contraction with the n-th component of the Chern character induces a derivation of the Gerstenhaber algebra $H^{\bullet}(X, \mathcal{T}_{\rm poly})$.* $\square$

Remark 8.3 Corollary 8.2 implies that the contractions with odd components of the Chern character are derivations of the cup product on $H^{\bullet}(X, \mathcal{T}_{\rm poly})$. This statement was formulated without a proof in [41, Theorem 9].

Remark 8.4 The Deligne-Drinfeld conjecture [25, Section 6] states that the Lie algebra $\mathfrak{grt}$ is freely generated by elements σ_n, n odd ≥ 3. Using a deep analysis of motivic multiple zeta values, F. Brown recently [7] proved that the Deligne-Drinfeld elements indeed generate a free Lie subalgebra in $\mathfrak{grt}$. However, it is still not known whether this subalgebra coincides with the full Lie algebra $\mathfrak{grt}$. If the Deligne-Drinfeld conjecture indeed holds then Theorem 8.1 gives an exhaustive geometric description of the action of $\mathfrak{grt}$ on the Gerstenhaber algebra $H^{\bullet}(X, \mathcal{T}_{\rm poly})$.

We will prove Theorem 8.1 in Section 8.1 and now we will present a combinatorial fact which is used in the proof of this theorem.

Claim 8.5 *Let Γ be a connected, 1-vertex irreducible labeled graph with $n+1$ vertices, such that each vertex of Γ has valency ≥ 3 and Γ does not have double edges. In addition, let Γ^{wheel}_n be the labeled graph depicted on figure 5.3. If Γ admits an orientation for which each vertex with label $\le n$ has at most one out-going edge, then $\Gamma = \sigma(\Gamma^{wheel}_n)$ for some permutation*[17] *$\sigma \in S_n$. Furthermore, Γ^{wheel}_n has exactly two orientations which satisfy the above condition. These orientations are shown on figures 8.1 and 8.2.*

Proof. *The vertex with label $(n+1)$ has no incoming edges.*

Let us assume that the vertex of Γ with label $(n+1)$ has an incoming edge which originates, say, at vertex i_1. Since vertex i_1 is at least trivalent and it has at most one

[16] In [1] and [25], the Lie algebra we use here is denoted by $\mathfrak{grt}_1$.

[17] The group S_n is tacitly identified with the stabilizer of $(n+1)$ in S_{n+1}.

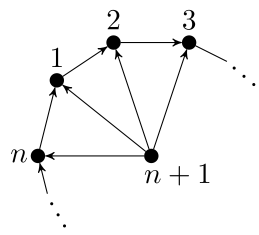

Figure 8.1: Each vertex with label $\leq n$ has at most 1 out-going edge

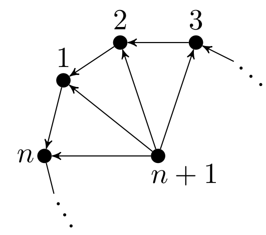

Figure 8.2: Each vertex with label $\leq n$ has at most 1 out-going edge

outgoing edge, this vertex has at least two incoming edges. One of these edges originates at, say, vertex i_2. Since Γ does not have double edges, $i_2 \leq n$ so we may apply the same argument to vertex i_2 and choose an edge which terminates at vertex i_2 and originates, say, at vertex i_3.

Continuing this process we will get an oriented path which returns to vertex $(n+1)$. Indeed, since the graph is finite and each vertex with label $\leq n$ cannot have more than one out-going edge, the path must come back to vertex $(n+1)$. See figure 8.3.

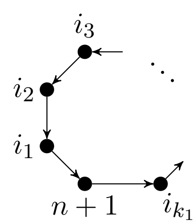

Figure 8.3: The oriented path returns to the vertex with label $(n+1)$

Since vertex i_{k_1} on figure (8.3) is at least trivalent and has at most one out-going edge, it has at least one more incoming edge which originates, say, at vertex j_1. Since Γ does not have double edges, $j_1 \leq n$. Hence, we can pick an edge which terminates at vertex j_1 and originates, say, at vertex j_2. We continue this process and get another oriented path which is shown on figure 8.4.

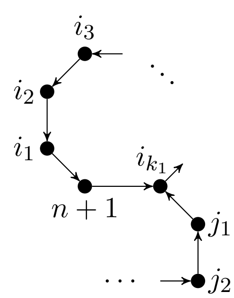

Figure 8.4: The oriented path returns to the vertex with label $(n+1)$

The path

$$i_{k_1} \leftarrow j_1 \leftarrow j_2 \leftarrow \dots \tag{8.2}$$

cannot arrive at any of vertices $i_1, i_2, \dots, i_{k_1}$ because vertices with labels $\leq n$ have at most one out-going edge. For the same reason, it cannot return to any of the vertices $j_1, j_2, \dots$. Hence path (8.2) will eventually return to the vertex with label $(n+1)$ and we get another oriented path from vertex $(n+1)$ to vertex i_{k_1}. This path is shown on figure 8.5.

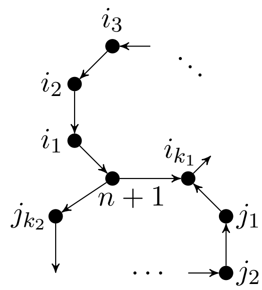

Figure 8.5: We found another oriented path from vertex i_{k_1} to vertex $(n+1)$

Let us now observe that $j_{k_2} \neq i_{k_1}$. Hence, applying the above argument once again we construct another oriented path which starts at vertex $(n+1)$, terminates at vertex j_{k_2}, and has length > 1.

This process of building oriented paths will not terminate and this contradicts to the fact that the graph Γ is finite.

Thus the vertex with label $(n+1)$ does not have incoming edges.

A vertex with label $i \leq n$ cannot have two incoming edges which originate at vertices with labels $\leq n$.

Indeed, let us consider an edge which originates, say, at vertex $i_1 \leq n$ and terminates at i. Then vertex i_1 has at least two incoming edges. At least one of these edges originates at vertex $i_2 \leq n$. We pick this edge and find an edge which terminates at i_2 and originates at a vertex with label $i_3 \leq n$.

Continuing this process we find an oriented path which goes only through vertices with labels $\leq n$ and terminates at i. Since the graph Γ is finite, we can complete the path

$$i \leftarrow i_1 \leftarrow i_2 \leftarrow i_3 \leftarrow \dots$$

to the cycle:

$$i \leftarrow i_1 \leftarrow i_2 \leftarrow i_3 \leftarrow \dots \leftarrow i_k \leftarrow i \tag{8.3}$$

with $i, i_1, i_2, \dots, i_k \leq n$.

If there is another edge which terminates at vertex i and originates, say, at vertex $j_1 \leq n$ then we may repeat the same process and find another oriented path which terminates at i and goes only through vertices with labels $\leq n$:

$$i \leftarrow j_1 \leftarrow j_2 \leftarrow j_3 \leftarrow \dots \tag{8.4}$$

Since each vertex with label $\le n$ has at most one out-going edge, the set of vertices

$$\{j_1, j_2, j_3, \dots\}$$

must have the empty intersection with the set of vertices in the cycle (8.3). In addition, the path (8.4) cannot return to any of the vertices $j_1, j_2, j_3, \dots$. This observation contradicts to the fact that the graph Γ is finite.

Thus, a vertex with label $i \le n$ cannot have two incoming edges which originate at vertices with labels $\le n$.

On the hand, every vertex with label $i \le n$ has at least two incoming edges and at most one out-going edge. Therefore every vertex with label $i \le n$ has valency 3. It has exactly one out-going edge which terminates at a vertex with label $\le n$; it has exactly one incoming edge which originates at a vertex with label $\le n$; and it has exactly one incoming edge which originates at the vertex with label $(n+1)$.

We conclude that the graph Γ is a "join of wheels" shown on figure 8.6.

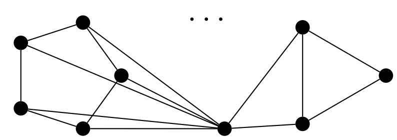

Figure 8.6: All unlabeled vertices on the picture should carry labels $\le n$

Since Γ is 1-vertex irreducible, we conclude immediately that $\Gamma = \sigma(\Gamma_n^{wheel})$ for some permutation $\sigma \in S_n$.

It is obvious that the orientations shown on figures 8.1 and 8.2 are the only possible orientations of Γ_n^{wheel} satisfying the condition stated in the claim.

Claim 8.5 is proved. $\square$

8.1 The proof of Theorem 8.1

Let Γ be an element of gra_{n+1} whose underlying graph is connected, 1-vertex irreducible, and each vertex of Γ has valency ≥ 3.

Then, for a local section v of the sheaf $\mathcal{FR}'$ (6.3) we consider the local section

$$\Gamma(\underbrace{\omega, \dots, \omega}_{n \text{ times}}, v) \tag{8.5}$$

of $\mathcal{FR}'$, where ω is the global section of the sheaf $\Omega^1(\mathcal{O}_X^{\mathrm{coord}})\hat{\otimes}T^{1,0}(P)$ introduced in Theorem 2.12 and the action of $\mathsf{Gra}(n)$ on $\mathcal{FR}'$ is obtained via extending $\mathfrak{a}$ (5.15) by linearity over $\Omega^{\bullet}(\mathcal{O}_X^{\mathrm{coord}})$.

Let us prove that

Claim 8.6 *The section* (8.5) *is zero unless there exists a permutation $\sigma \in S_n$ such that the underlying labeled graph for Γ equals $\sigma(\Gamma_n^{wheel})$, where Γ_n^{wheel} is shown on figure 5.3.*

Proof. Recall that an action of Γ on a collection of polyvector fields is expressed in terms of the operators (5.4).

So we will keep track of terms involving the sum

$$\sum_{a=1}^{d} 1\otimes\cdots\otimes 1\otimes \underbrace{\partial_{\xi_a}}_{i\text{-th slot}} \otimes 1\otimes\cdots\otimes 1\otimes \underbrace{\partial_{x^a}}_{j\text{-th slot}} \otimes 1\otimes\cdots\otimes 1$$

by choosing a direction on the edge (i,j) from vertex i to vertex j. Similarly, we will keep track of terms involving the sum

$$\sum_{a=1}^{d} 1\otimes\cdots\otimes 1\otimes \underbrace{\partial_{x^a}}_{i\text{-th slot}} \otimes 1\otimes\cdots\otimes 1\otimes \underbrace{\partial_{\xi_a}}_{j\text{-th slot}} \otimes 1\otimes\cdots\otimes 1$$

by choosing a direction on the edge (i,j) from vertex j to vertex i.

We see that $\Gamma(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v)$ splits into the sum over all possible orientations of Γ and, since ω is vector-valued, a summand corresponding a given orientation of Γ is zero if this orientation does not satisfy this property: *each vertex of Γ with label $\le n$ has at most one out-going edge.*

Thus Claim 8.5 implies the desired statement. $\square$

Let us now recall that the class $\widetilde{\sigma}_n\in H^0(\mathsf{GC})$ can be represented by a cocycle γ_n of the form

$$\gamma_n=\lambda\sum_{\sigma\in S_{n+1}}\sigma(\Gamma_n^{wheel})+\sum_{i=1}^{k}\lambda_i\Gamma_i\in \mathbf{s}^{2n}\big(\mathsf{Gra}(n+1)\big)^{S_{n+1}}\,, \tag{8.6}$$

where λ,λ_i are non-zero scalars, Γ_n^{wheel} is the graph shown on figure 5.3 and for each index i the underlying unlabeled graph Γ_i is not isomorphic to Γ_n^{wheel}. In addition, every graph Γ_i is connected, 1-vertex irreducible and each vertex of Γ_i has valency ≥ 3.

Let v be a cocycle in $\check{C}^\bullet(X,\mathcal{FR})$ which represents a cohomology class in (8.1).

Then the cohomology class $\mathsf{D}_{\gamma_n}([v])$ is represented by the Čech cocycle w^n with

$$w^n_{\alpha_0\dots\alpha_m}:=\frac{\lambda}{n!}\sum_{\sigma\in S_{n+1}}\sigma(\Gamma_n^{wheel})(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v_{\alpha_0\dots\alpha_m})\;+\;\sum_{i=1}^{k}\frac{\lambda_i}{n!}\Gamma_i(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v_{\alpha_0\dots\alpha_m})\,. \tag{8.7}$$

Claim 8.6 implies that

$$\Gamma_i(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v_{\alpha_0\dots\alpha_m})=0$$

for all i and

$$\sigma(\Gamma_n^{wheel})(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v_{\alpha_0\dots\alpha_m})=0$$

unless $\sigma(n+1)=n+1$.

Thus,

$$w^n_{\alpha_0\dots\alpha_m}=\frac{\lambda}{n!}\sum_{\sigma\in S_n}\sigma(\Gamma_n^{wheel})(\underbrace{\omega,\dots,\omega}_{n\text{ times}},v_{\alpha_0\dots\alpha_m}) \tag{8.8}$$

or equivalently,

$$w^n_{\alpha_0\dots\alpha_m} = \lambda\,\Gamma^{wheel}_n(\underbrace{\omega,\dots,\omega}_{n \text{ times}}, v_{\alpha_0\dots\alpha_m})\,. \tag{8.9}$$

Claim 8.5 implies that an orientation of Γ^{wheel}_n gives the zero contribution to the right hand side of (8.9), unless this is the orientation shown on figure 8.1 or the orientation shown on figure 8.2.

Using this observation, it is not hard to see that $w^n_{\alpha_0\dots\alpha_m}$ is obtained by contracting $v_{\alpha_0\dots\alpha_m}$ with the global section

$$\lambda' \sum_{1\le a_1,\dots,a_n\le d}\ \sum_{1\le b_1,\dots,b_n\le d} \frac{\partial^2\omega^{a_1}}{\partial t^{a_2}\partial t^{b_1}}\frac{\partial^2\omega^{a_2}}{\partial t^{a_3}\partial t^{b_2}}\cdots\frac{\partial^2\omega^{a_n}}{\partial t^{a_1}\partial t^{b_n}} dt^{b_1}\cdots dt^{b_n} \tag{8.10}$$

of the sheaf $\Omega^\bullet(\mathcal{O}^{\rm coord}_X)\,\hat{\otimes}\,\Omega^n_{\mathbb{K}}(P)$, where λ' is a non-zero scalar.

By Theorem 4.1, the global section

$$-\frac{1}{n!} \sum_{1\le a_1,\dots,a_n\le d}\ \sum_{1\le b_1,\dots,b_n\le d} \frac{\partial^2\omega^{a_1}}{\partial t^{a_2}\partial t^{b_1}}\frac{\partial^2\omega^{a_2}}{\partial t^{a_3}\partial t^{b_2}}\cdots\frac{\partial^2\omega^{a_n}}{\partial t^{a_1}\partial t^{b_n}} dt^{b_1}\cdots dt^{b_n} \tag{8.11}$$

represents the n-th component of the Chern character on X.

Thus, using Theorem 3.3, we conclude that the contraction of the class $[v]$ of v with the n-th component of the Chern character on X is indeed proportional to $\mathsf{D}_{\gamma_n}([v])$ (with a non-zero coefficient).

Theorem 8.1 is proved. $\square$

9 Application: Isomorphisms between harmonic and Hochschild structures

Let X be a smooth algebraic variety (over an algebraically closed field of characteristic zero). Let, as above, $\mathcal{T}_{\rm poly}$ be the sheaf of polyvector fields on X and $C^\bullet(\mathcal{O}_X)$ be the sheaf of polydifferential operators on X .

Following [14], we call the Gerstenhaber algebras

$$H^\bullet(X, \mathcal{T}_{\rm poly}) \tag{9.1}$$

and

$$H^\bullet(X, C^\bullet(\mathcal{O}_X)) \tag{9.2}$$

the *harmonic structure* and the *Hochschild structure* of X, respectively.

Due to the Hochschild-Kostant-Rosenberg (HKR) theorem, the canonical embedding

$$\mathcal{T}_{\rm poly} \hookrightarrow C^\bullet(\mathcal{O}_X) \tag{9.3}$$

induces an isomorphism of Gerstenhaber algebras from (9.1) to (9.2), provided X is affine.

In general, the HKR map (9.3) does not induce an isomorphism of Gerstenhaber algebras from (9.1) to (9.2). However, according to [13, Theorem 1.3], the correction of the HKR map

by the "square root of" the Todd class of X does induce an isomorphism[18] of Gerstenhaber algebras (9.1) and (9.2).

Let us recall that the "square root of" the Todd class of X is given by the formula:

$$\mathrm{Td}^{1/2}(X) = \det(\widetilde{q}([\mathsf{A}]))\,, \tag{9.4}$$

where

$$\widetilde{q}(t) = \left(\frac{t}{1-e^{-t}}\right)^{1/2} \tag{9.5}$$

and $[\mathsf{A}]$ denotes the Atiyah class of X.

It was proved in [13] that the correction of the HKR map (9.3) by the $\hat{A}$-genus

$$\hat{A}(X) = \det(q([\mathsf{A}]))\,, \qquad q(t) = \left(\frac{t}{e^{t/2}-e^{-t/2}}\right)^{1/2} \tag{9.6}$$

also induces an isomorphism of Gerstenhaber algebras (9.1) and (9.2).

Let us observe that the function $q(t)$ is even and the function $\widetilde{q}(t)$ is related to $q(t)$ by the formula

$$\log(q(t)) = \frac{1}{2}\big(\log(\widetilde{q}(t)) + \log(\widetilde{q}(-t))\big)\,. \tag{9.7}$$

This observation motivates us to introduce the notion of generalized $\hat{A}$-genus:

Definition 9.1 *Let $f(t)$ be a formal power series in $1+t\mathbb{K}[[t]]$ for which the even part of $\log(f(t))$ coincides with the Taylor series of the function*

$$\frac{1}{2}\log\left(\frac{t}{e^{t/2}-e^{-t/2}}\right) \tag{9.8}$$

at $t=0$. Then the generalized $\hat{A}$-genus *$\hat{A}_f(X)$ of X corresponding to the series f is defined by the equation*

$$\hat{A}_f(X) = \det(f([\mathsf{A}]))\,, \tag{9.9}$$

where $[\mathsf{A}]$ is the Atiyah class of X.

Theorem 8.1 implies the following remarkable statement:

Theorem 9.2 *Let X be a smooth algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristic zero and let $f(t)$ be a formal power series in $1+t\mathbb{K}[[t]]$ for which the even part of $\log(f(t))$ coincides with the Taylor expansion of (9.8). Then the correction of the HKR map by the generalized $\hat{A}$-genus $\hat{A}_f(X)$ induces an isomorphism of Gerstenhaber algebras (9.1) and (9.2).*

Proof. Let us recall that the n-th component $c_n(X)$ of the Chern character of X is represented by

$$\frac{1}{n!}\mathrm{tr}\,\mathsf{A}^n\,,$$

[18]The existence of such an isomorphism also follows from the results of [22].

where A is any representative of the Atiyah class. To prove Theorem 9.2, it suffices to show that for every odd $n \geq 1$

$$\exp\left(\frac{1}{n!}\mathrm{tr}\,\mathsf{A}^n\right)$$

induces an automorphism of the Gerstenhaber algebra (9.1).

For $n = 1$ this fact is proved in [13, Section 10.3] and the remaining cases n odd ≥ 3 are covered by Theorems 7.1 and 8.1. □

Remark 9.3 This statement is very similar to Proposition 6.2 in [1]. We suspect, based on this, that there may exist a deep link between solutions of the generalized Kashiwara-Vergne problem [1] and the above isomorphisms between harmonic and Hochschild structures on an algebraic variety. It is likely that this link can be found using the ideas developed in [11], [12], [15], [39], and [43].

10 Examples

It is not easy to find examples of varieties X for which the sheaf cohomology of the sheaf of polyvector fields has been computed explicitly in the literature. Furthermore, the authors do not know any general tools to determine the Gerstenhaber algebra structure on $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$. This is unfortunate, since this Gerstenhaber algebra structure is an invariant of the variety, potentially containing valuable information.

Corollary 8.2 shall be seen as a first step towards the determination of this Gerstenhaber structure. It gives a very general constraint on the possible products and brackets.

Here we give some examples, for which at least $H^\bullet(X, \mathcal{T}_{\mathrm{poly}})$ may be computed explicitly. In this section, we assume that $\mathbb{K} = \mathbb{C}$ and we use freely tools of complex algebraic geometry. Our goal is to compute

$$H^q(X, \mathcal{T}^k_{\mathrm{poly}}) \cong H^q(X, \Omega^{d-k}_X \otimes K^{-1}_X) \tag{10.1}$$

where K_X is the canonical bundle of X and $d = \dim_{\mathbb{C}} X$.

1. For $\mathbb{P}^d$, Grassmannians, and some simple enough flag manifolds the sheaf cohomology of the polyvector fields can be deduced from the Borel-Weil-Bott theorem. Unfortunately, by the same theorem the cohomology is concentrated in degree $q = 0$ and the statement of Corollary 8.2 is trivial.

2. For Calabi-Yau varieties the canonical bundle K_X is trivial and the numbers $\dim H^q(X, \mathcal{T}^k_{\mathrm{poly}}) = h^{d-k,q}$ agree with the Hodge numbers of X.

3. For complete intersections in $\mathbb{P}^N$ there are explicit formulas for all the twisted Hodge numbers $h^{p,q}_j = \dim H^q(X, \Omega^p_X(j))$. They have been computed by P. Brückmann [10, Satz 3] (see also [8, 9]). Together with the adjunction formula it follows that $H^q(X, \mathcal{T}^k_{\mathrm{poly}})$ can be computed explicitly in this case.

We will focus on the latter class of examples. Since these results seem not so well known, let us sketch a possible way to compute the twisted Hodge diamond (i. e., the numbers $h^{p,q}_j$)

for smooth complete intersections $X = Y_1 \cap \cdots \cap Y_r \subset \mathbb{P}^{d+r}$. The twisted Hodge diamond has the following general form (see [10, Folgerung 2]):

$$
\begin{array}{ccccccccc}
 & & & & - & & & & \\
 & & & & \diamondsuit & - & & & \\
 & & & & \vdots & & \ddots & & \\
 & & & & \diamondsuit & & & - & \\
* & * & \cdots & * & * & * & \cdots & * & * \\
 & + & & & \diamondsuit & & & & \\
 & & \ddots & & \vdots & & & & \\
 & & & + & \diamondsuit & & & & \\
 & & & \nwarrow p & + & q \nearrow & & &
\end{array}
$$

The numbers not shown are all zero. The symbols $*$, $+$, $-$ stand for some possibly non-zero numbers.[19] The numbers $\diamondsuit$ are only present for $j = 0$. The numbers $+$ are zero for $j < 0$ and the numbers $-$ are zero for $j > 0$. In fact, by the weak Lefschetz Theorem, for $j = 0$ all the $+$ and $-$ are zero, except for the ones in the central column, which are 1. There are explicit formulas for all numbers $+$, $-$, $*$. To get them, one may proceed as follows. For $j = 0$ the explicit formula was given by Hirzebruch [35, Section 22]. It is a classical application of the Riemann-Roch-Hirzebruch Theorem. For $j < 0$ one may use the Serre duality to restrict to the case $j > 0$. For $j > 0$ the numbers $-$ vanish. The numbers $+$ are the numbers of global holomorphic sections, which are not difficult to compute. The remaining numbers $*$ can be computed from the explicit formula for the Euler characteristic of $H^\bullet(X, \Omega^p_X(j))$ given by Hirzebruch, see [35], eqn. (2) on page 160. The resulting expressions for the $h^{p,q}_j$ are explicit, but neither pretty nor relevant here, so we refer to [10, Satz 3] and [8, 9] instead.

Now let us consider polyvector fields on a complete intersection X. By the adjunction formula, $\deg K_X = -d - r - 1 + \sum_j d_j$, where d_j is the degree of Y_j. Hence, using (10.1), we see that

$$\dim H^q(X, \mathcal{T}^k_{\mathrm{poly}}) = h^{d-k,q}_{d+r+1-\sum_j d_j}.$$

10.1 The case of Calabi-Yau complete intersections

A smooth complete intersection X is Calabi-Yau if and only if

$$d + r + 1 - \sum_j d_j = 0\,.$$

[19] In particular not all $*$'s are the same in general etc.

In this case, the only nontrivial entries of the Hodge diamond are the numbers $*$, and the numbers $\diamondsuit$. The corresponding classes of polyvector fields live in $H^k(X, \mathcal{T}^k_{\mathrm{poly}})$ (corresponding to the $*$ entries) and $H^q(X, \mathcal{T}^{d-q}_{\mathrm{poly}})$ (corresponding to the $\diamondsuit$ entries). Looking at the Hodge diamond, one sees that for even d all Lie brackets between such classes must vanish. However, for $d = 2n+1$, one has the following potentially non-vanishing brackets

$$[,]\colon H^k(X, \mathcal{T}^k_{\mathrm{poly}}) \times H^n(X, \mathcal{T}^{n+1}_{\mathrm{poly}}) \to H^{n+k}(X, \mathcal{T}^{n+k}_{\mathrm{poly}}) \tag{10.2}$$

$$[,]\colon H^k(X, \mathcal{T}^k_{\mathrm{poly}}) \times H^{n+1-k}(X, \mathcal{T}^{n+1-k}_{\mathrm{poly}}) \to H^{n+1}(X, \mathcal{T}^{n}_{\mathrm{poly}}). \tag{10.3}$$

Next consider the wedge products. One has the following potentially non-trivial components.

$$\begin{aligned} &\wedge\colon H^k(X, \mathcal{T}^k_{\mathrm{poly}}) \times H^{k'}(X, \mathcal{T}^{k'}_{\mathrm{poly}}) \to H^{k+k'}(X, \mathcal{T}^{k+k'}_{\mathrm{poly}}) \\ &\wedge\colon H^q(X, \mathcal{T}^{d-q}_{\mathrm{poly}}) \times H^{q'}(X, \mathcal{T}^{d-q'}_{\mathrm{poly}}) \to H^{d}(X, \mathcal{T}^{d}_{\mathrm{poly}}) \cong \mathbb{C} \qquad \text{for } q+q'=d. \end{aligned}$$

Here we consider the product with $1 \in H^0(X, \mathcal{T}^0_{\mathrm{poly}})$ as a trivial operation.

Let us now show that among Calabi-Yau complete intersections we have plenty of examples X for which the odd components $ch_{2l+1}(X)$, $l \geq 1$ of the Chern character are non-zero and they act non-trivially on the cohomology

$$H^\bullet(X, \mathcal{T}_{\mathrm{poly}})\,. \tag{10.4}$$

For this purpose, we observe that the n-th component $ch_n(X)$ of the Chern character $ch(X)$ can be expressed in terms of the first Chern class h of the hyperplane bundle. Namely,

$$ch_n(X) = (d + r + 1 - \sum_j d_j^n)\frac{h^n}{n!} \ \in \ H^n(X, \Omega^n)\,. \tag{10.5}$$

For generic values of $d, r, d_1, \ldots, d_r$, the coefficients $(d + r + 1 - \sum_j d_j^n)$ are non-zero if $n > 1$ and the classes h^n come from algebraic cycles on X.

Next, we remark that vectors in $H^q(X, \mathcal{T}^{d-q}_{\mathrm{poly}})$, $q = 0, 1, \ldots$ correspond to Hodge classes of X and we have plenty of such non-trivial classes also coming from algebraic cycles on X.

On the other hand, if classes $\eta \in H^n(X, \Omega^n)$ and $v \in H^q(X, \mathcal{T}^{d-q}_{\mathrm{poly}})$ come from algebraic cycles η' and v' on X then the contraction of η with v gives us a class in $H^{q+n}(X, \mathcal{T}^{d-q-n}_{\mathrm{poly}})$ which corresponds intersection of the cycles η' and v'. Hence this class is non-trivial provided $n + q \leq d$.

Thus we see that for generic values of $d, r, d_1, \ldots, d_r$, the components $ch_n(X)$, $n > 1$ of the Chern character of X act non-trivially on (10.4).

Combining these considerations with Theorem 8.1, we see that Calabi-Yau complete intersections provide us with a large supply of non-trivial representations of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$.

Using Corollary 8.2, we also deduce the following interesting statements:

- Let $\gamma \in H^k(X, \mathcal{T}^k_{\mathrm{poly}})$, $\nu \in H^{k'}(X, \mathcal{T}^{k'}_{\mathrm{poly}})$ with $k + k' = d$ and $k, k' > 0$. Then the contraction of $ch_{2l+1}(X)$ with $\gamma \wedge \nu$ is zero.

- If d is odd and $\gamma \in H^k(X, \mathcal{T}^k_{\text{poly}})$, $\nu \in H^{k'}(X, \mathcal{T}^{k'}_{\text{poly}})$ are such that $2k + 2k' - 1 = d$, then the contraction of $ch_{2l+1}(X)$ with $[\gamma, \nu] \in H^{(d+1)/2}(X, \mathcal{T}^{(d-1)/2}_{\text{poly}})$ vanishes ($l = 0, 1, 2, \dots$). This implies that $[\gamma, \nu] = 0$.

- Similarly, one may show that the operations (10.2) vanish as well.

Hence the (only) "interesting" algebraic operation in $H^\bullet(X, \mathcal{T}_{\text{poly}})$ is the product on the sub-algebra

$$A = \bigoplus_k H^k(X, \mathcal{T}^k_{\text{poly}}). \tag{10.6}$$

10.2 The case of Fano complete intersections

We now consider the case $d + r + 1 - \sum_j d_j > 0$, i.e., X being a Fano variety[20]. In this case the numbers $-$ are all zero, while the numbers $+$ and $*$ in general are not. The Gerstenhaber structure on $H^\bullet(X, \mathcal{T}_{\text{poly}})$ reduces to the following data: First, we again have the commutative sub-algebra (10.6). Second, holomorphic polyvector fields form the Gerstenhaber sub-algebra

$$B := \bigoplus_k H^0(X, \mathcal{T}^k_{\text{poly}})\,.$$

The only potentially nontrivial Gerstenhaber operations between elements of A and B (counting the product with 1 as trivial) are the Lie brackets of elements of $H^0(X, \mathcal{T}^1_{\text{poly}})$ with elements of A.

Unfortunately, in this case, contractions with the odd components $ch_{2l+1}(X)$, $l \geq 1$, of the Chern character are trivial.

A Homotopy O-algebras. Deformation complex of an O-algebra

Let O be an operad (possibly with a non-zero differential). We assume that O admits a cobar resolution

$$\varphi_O : \mathrm{Cobar}(C) \xrightarrow{\sim} O\,, \tag{A.1}$$

where C is a coaugmented dg cooperad satisfying the following technical condition: *the cokernel $C_\circ$ of the coaugmentation carries an ascending filtration*

$$\mathbf{0} = \mathcal{F}^0 C_\circ \subset \mathcal{F}^1 C_\circ \subset \mathcal{F}^2 C_\circ \subset \dots \tag{A.2}$$

which is compatible with the pseudo-operad structure on $C_\circ$, and

$$C_\circ(n) = \bigcup_m \mathcal{F}^m C_\circ(n)\,, \qquad \forall \;\; n \geq 0\,. \tag{A.3}$$

[20]The case of $d + r + 1 - \sum_j d_j < 0$ can also be considered, of course, but it adds nothing new to the discussion.

For example, if the dg cooperad C has the properties

$$C(1) \cong \mathbb{K}, \qquad\qquad C(0) = \mathbf{0} \tag{A.4}$$

then the filtration "by arity" on $C_\circ$ satisfies the above technical condition.

In this paper, we mostly use $O = \mathsf{Ger}$ or $O = \Lambda\mathsf{Lie}$. In the former case, C is the linear dual to $\Lambda^{-2}\mathsf{Ger}$ and in the latter case $C = \Lambda^2\mathsf{coCom}$. It is clear that, in both cases, condition (A.4) is satisfied.

Recall that, for a cochain complex $\mathcal{V}$, we denote by

$$C(\mathcal{V}) := \bigoplus_{n \ge 1} \Big(C(n) \otimes \mathcal{V}^{\otimes\, n} \Big)^{S_n} \tag{A.5}$$

the "cofree" C-coalgebra co-generated by $\mathcal{V}$.

We also denote by

$$\mathsf{coDer}\big(C(\mathcal{V})\big) \tag{A.6}$$

the cochain complex of coderivations of the C-coalgebra $C(\mathcal{V})$.

In other words, $\mathsf{coDer}\big(C(\mathcal{V})\big)$ consists of $\mathbb{K}$-linear maps

$$\mathcal{D} : C(\mathcal{V}) \to C(\mathcal{V}) \tag{A.7}$$

which are compatible with the C-coalgebra structure on $C(\mathcal{V})$ in the following sense:

$$\Delta_n \circ \mathcal{D} = \sum_{i=1}^{n} \big(\mathrm{id}_C \otimes \mathrm{id}_{\mathcal{V}}^{\otimes(i-1)} \otimes \mathcal{D} \otimes \mathrm{id}_{\mathcal{V}}^{n-i} \big) \circ \Delta_n \tag{A.8}$$

where Δ_n is the comultiplication map

$$\Delta_n : C(\mathcal{V}) \to \Big(C(n) \otimes \big(C(\mathcal{V})\big)^{\otimes n} \Big)^{S_n}.$$

The $\mathbb{Z}$-graded vector space (A.6) carries the natural differential ∂ which comes from those on C and $\mathcal{V}$.

Since the commutator of two coderivations is again a coderivation, the cochain complex (A.6) is naturally a dg Lie algebra.

We denote by

$$\mathsf{coDer}'\big(C(\mathcal{V})\big) \tag{A.9}$$

the dg Lie subalgebra of coderivations $\mathcal{D} \in \mathsf{coDer}\big(C(\mathcal{V})\big)$ satisfying the additional technical condition

$$\mathcal{D}\Big|_{\mathcal{V}} = 0\,. \tag{A.10}$$

Recall that, since the C-coalgebra $C(\mathcal{V})$ is cofree, every coderivation $\mathcal{D} : C(\mathcal{V}) \to C(\mathcal{V})$ is uniquely determined by it composition $p_{\mathcal{V}} \circ \mathcal{D}$ with the canonical projection:

$$p_{\mathcal{V}} : C(\mathcal{V}) \to \mathcal{V}\,. \tag{A.11}$$

It is not hard to see that the map

$$\mathcal{D} \mapsto p_{\mathcal{V}} \circ \mathcal{D}$$

induces isomorphisms of dg Lie algebras

$$\mathsf{coDer}\big(C(\mathcal{V})\big) \cong \mathrm{Conv}(C, \mathrm{End}_{\mathcal{V}})\,, \tag{A.12}$$

and

$$\mathsf{coDer}'\big(C(\mathcal{V})\big) \cong \mathrm{Conv}(C_{\circ}, \mathrm{End}_{\mathcal{V}})\,, \tag{A.13}$$

where the differential ∂ on $\mathrm{Conv}(C, \mathrm{End}_{\mathcal{V}})$ and $\mathrm{Conv}(C_{\circ}, \mathrm{End}_{\mathcal{V}})$ comes solely from the differential on C and $\mathcal{V}$.

Recall that [21, Proposition 5.2] $\mathrm{Cobar}(C)$-algebra structures on a cochain complex $\mathcal{V}$ are in bijection with degree 1 coderivations

$$Q \in \mathsf{coDer}'\big(C(\mathcal{V})\big) \tag{A.14}$$

satisfying the Maurer-Cartan equation

$$\partial Q + \frac{1}{2}[Q, Q] = 0\,. \tag{A.15}$$

Hence, given a $\mathrm{Cobar}(C)$-algebra structure on $\mathcal{V}$, we may consider the dg Lie algebras (A.12), (A.13) and the C-coalgebra $C(\mathcal{V})$ with the new differentials

$$\partial + [Q,\ ]\,, \tag{A.16}$$

and

$$\partial + Q\,, \tag{A.17}$$

respectively.

Any dg O-algebra $\mathcal{V}$ is naturally a $\mathrm{Cobar}(C)$-algebra. Thus, any O-algebra structure on $\mathcal{V}$ gives us a Maurer-Cartan element (A.14) and hence the new differential (A.16) on

$$\mathsf{coDer}\big(C(\mathcal{V})\big) \cong \mathrm{Conv}(C, \mathrm{End}_{\mathcal{V}})\,. \tag{A.18}$$

Definition A.1 *The cochain complex* (A.18) *with the differential* (A.16) *is called the* deformation complex *of the O-algebra $\mathcal{V}$. We denote this complex by $\mathbf{Def}_{O}(\mathcal{V})$ or simply $\mathbf{Def}(\mathcal{V})$ when the operad O is clear from the context.*

For more details about the deformation complex and its properties we refer the reader to papers [23] and [44].

For example, if $O = \Lambda\mathsf{Lie}$ then, we may choose $C = \Lambda^2\mathsf{coCom}$ and, in this case, $\mathbf{Def}_{O}(\mathcal{V})$ is the truncated version of the Chevalley-Eilenberg cochain complex of $\mathcal{V}$ with coefficients in $\mathcal{V}$.

It turns out that the deformation complex is a homotopy invariant of an O-algebra. More precisely,

Theorem A.2 *If dg O-algebras $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic then the dg Lie algebra $\mathbf{Def}_{O}(\mathcal{A})$ is quasi-isomorphic to the dg Lie algebra $\mathbf{Def}_{O}(\mathcal{B})$.*

Proof. This statement is proved in a wider generality in [23]. □

Remark A.3 The construction of the deformation complex $\mathbf{Def}_O(\mathcal{V})$ depends on the choice of the cooperad C in (A.1). However, using homological properties [21, Section 4.4] of the bi-functor Conv, it is not hard to prove that, if dg cooperads C and $\widetilde{C}$ are quasi-isomorphic, then the dg Lie algebras $\mathrm{Conv}(C, \mathrm{End}_{\mathcal{V}})$ and $\mathrm{Conv}(\widetilde{C}, \mathrm{End}_{\mathcal{V}})$ are also quasi-isomorphic. Here, the dg Lie algebras $\mathrm{Conv}(C, \mathrm{End}_{\mathcal{V}})$ and $\mathrm{Conv}(\widetilde{C}, \mathrm{End}_{\mathcal{V}})$ are considered with the differentials coming from the O-algebra structure on $\mathcal{V}$.

A.1 A cocycle in $\mathbf{Def}_O(\mathcal{V})$ induces a derivation of the O-algebra $H^\bullet(\mathcal{V})$

Let O be an augmented operad in the category of graded vector spaces and $\mathcal{V}$ be a dg O-algebra. In this subsection, we show that any cocycle in the deformation complex $\mathbf{Def}_O(\mathcal{V})$ induces a derivation of the O-algebra $H^\bullet(\mathcal{V})$.

Let $\mathcal{D}$ be a cochain in the deformation complex $\mathbf{Def}_O(\mathcal{V})$ and v be a cochain in $\mathcal{V}$.

We claim that

Proposition A.4 *The equation*

$$\mathfrak{B}_{\mathcal{D}}(v) := \mathcal{D}(v) \tag{A.19}$$

defines a chain map

$$\mathfrak{B} : \mathbf{Def}_O(\mathcal{V}) \to \mathrm{Hom}(\mathcal{V}, \mathcal{V})\,. \tag{A.20}$$

For any cocycle $\mathcal{D}$ in $\mathbf{Def}_O(\mathcal{V})$ the induced map

$$H^\bullet(\mathcal{V}) \to H^\bullet(\mathcal{V}) \tag{A.21}$$

is a derivation of the O-algebra $H^\bullet(\mathcal{V})$.

Proof. The compatibility of $\mathfrak{B}$ with the differentials follows directly from definitions.

To prove that (A.21) is a derivation, we recall that O receives a quasi-isomorphism φ_O (A.1) from $\mathrm{Cobar}(C)$.

Due to Remark A.3, we have a freedom of choosing a convenient Cobar-resolution of O. So we choose $C = \mathrm{Bar}(O)$ and observe that every vector β in $O_\circ(n)$ gives us a cocycle

$$\beta' := \mathbf{s}\,(\mathbf{s}^{-1}\,\beta) \in \mathrm{Cobar}\big(\mathrm{Bar}(O)\big) \tag{A.22}$$

which satisfies the property

$$\varphi_O(\beta') = \beta\,. \tag{A.23}$$

Next, for every n-tuple of cocycles

$$v_1, v_2, \dots, v_n \in \mathcal{V}$$

we consider the cocycle

$$(\mathbf{s}^{-1}\,\beta; v_1, v_2, \dots, v_n) \tag{A.24}$$

in the "cofree" $\mathrm{Bar}(O)$-coalgebra $\mathrm{Bar}(O)(\mathcal{V})$.

Since the coderivation $\mathcal{D}$ is closed with respect to the differential $\partial + [Q, \ ]$, we conclude that

$$\partial \circ \mathcal{D}(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) + Q \circ \mathcal{D}(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) \tag{A.25}$$
$$-(-1)^{|\mathcal{D}|}\mathcal{D} \circ Q(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) = 0\,.$$

Using the fact that for every elementary co-insertion

$$\Delta_{i,q} : \mathrm{Bar}(O)(n) \to \mathrm{Bar}(O)(n-q+1) \otimes \mathrm{Bar}(O)(q)$$

$$\Delta_{i,q}(\mathbf{s}^{-1}\beta) = 0\,,$$

we deduce that

$$\mathcal{D}(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) = p_{\mathcal{V}} \circ \mathcal{D}(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) \tag{A.26}$$

$$+ \sum_{i=1}^{n} (-1)^{|\mathcal{D}|(|\beta|-1+|v_1|+\dots+|v_{i-1}|)}(\mathbf{s}^{-1}\beta; v_1, \dots, v_{i-1}, \mathcal{D}(v_i), v_{i+1}, \dots, v_n)\,,$$

and

$$Q(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) = p_{\mathcal{V}} \circ Q(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n) = \beta(v_1, v_2, \dots, v_n)\,. \tag{A.27}$$

Thus, applying the projection $p_{\mathcal{V}}$ to both sides of (A.25) and moving terms around, we get

$$\mathcal{D} \circ \beta(v_1, v_2, \dots, v_n) - \sum_{i=1}^{n} (-1)^{|\mathcal{D}|(|\beta|+|v_1|+\dots+|v_{i-1}|)}\beta(v_1, \dots, v_{i-1}, \mathcal{D}(v_i), v_{i+1}, \dots, v_n) = \tag{A.28}$$

$$(-1)^{|\mathcal{D}|}\partial\big(p_{\mathcal{V}} \circ \mathcal{D}(\mathbf{s}^{-1}\beta; v_1, v_2, \dots, v_n)\big)\,.$$

Proposition A.4 is proved. □

B Sheaves of algebras over an operad

B.1 Reminder of the Thom-Sullivan normalization

Let Δ be the simplicial category. In other words, objects of Δ are ordered sets $[n] := \{0 < 1 < \dots < n\}$, $n \ge 0$ and morphisms are non-decreasing functions from

$$f : \{0 < 1 < \dots < k\} \to \{0 < 1 < \dots < n\}\,.$$

For the geometric n-simplex

$$\mathbf{\Delta}_n = \{(u_0, u_1, \dots, u_n) \in \mathbb{R}^{n+1} \quad | \quad u_i \ge 0, \quad u_0 + u_1 + \dots + u_n = 1\} \tag{B.1}$$

we denote by $C^{\bullet}_{simp}(\mathbf{\Delta}_n)$ the normalized simplicial cochain complex and by $\Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_n)$ the dg commutative algebra of polynomial exterior forms on $\mathbf{\Delta}_n$. Both with coefficients in $\mathbb{K}$. It is easy to see that the collection

$$C^{\bullet}_{simp}(\mathbf{\Delta}_n) \tag{B.2}$$

is a simplicial object in the category of dg associative algebras over $\mathbb{K}$ and

$$\Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_n) \tag{B.3}$$

is a simplicial object in the category of dg commutative algebras over $\mathbb{K}$.

The formal integration of polynomial exterior forms gives us a natural map of cosimplicial objects

$$\mathfrak{I}_* : \Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_*) \to C^{\bullet}_{simp}(\mathbf{\Delta}_*) \tag{B.4}$$

and the Stokes theorem implies that this map is compatible with the differentials. Furthermore, [6, Proposition 3.3] implies that there exists a sequence of maps ($n \ge 0$)

$$\chi_n : \Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_n) \otimes \Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_n) \to C^{\bullet}_{simp}(\mathbf{\Delta}_n) \tag{B.5}$$

of degree -1 which are compatible with faces and degeneracies, and

$$\mathfrak{I}_n(\eta_1\eta_2) - \mathfrak{I}_n(\eta_1)\,\mathfrak{I}_n(\eta_2) = d\chi_n(\eta_1,\eta_2) + \chi_n(d\eta_1,\eta_2) + (-1)^{|\eta_1|}\chi_n(\eta_1, d\eta_2) \tag{B.6}$$

for all $\eta_1, \eta_2 \in \Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_n)$. In particular, it means that the multiplication on (B.2) is commutative up to homotopy.

Let us now consider a topological space X with a fixed locally finite cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ by open subsets. Then to any dg sheaf $\mathcal{V}$ on X, we may assign the cosimplicial cochain complex $\check{C}(\mathcal{V})$ whose n-th level is

$$\check{C}(\mathcal{V})_n := \prod_{\alpha_0,\dots,\alpha_n} \Gamma(U_{\alpha_0}\cap\dots\cap U_{\alpha_n}, \mathcal{V})\,.$$

The i-th co-face comes from omitting U_{α_i} and the i-th degeneracy comes from repeating U_{α_i} twice.

It is not hard to see that $\check{C}$ is a functor from the category of dg sheaves on X to the category of cosimplicial cochain complexes.

The normalized Čech complex of a dg sheaf $\mathcal{V}$ can be defined as

$$\check{C}^{\bullet}(\mathcal{V}) := C^{\bullet}_{simp}(\mathbf{\Delta}_*) \otimes_{\mathbf{\Delta}} \check{C}(\mathcal{V})_*\,. \tag{B.7}$$

In other words, $\check{C}^{\bullet}(\mathcal{V})$ is the subspace of

$$\prod_{n\ge 0} C^{\bullet}_{simp}(\mathbf{\Delta}_n) \otimes \check{C}(\mathcal{V})_n$$

which consists of sums

$$\sum_{n\ge 0} \kappa_n \otimes w_n$$

satisfying the following condition: *for every* $\varphi : [k] \to [n]$ $\kappa_n \otimes \varphi_*(w_k) = \varphi^*(\kappa_n) \otimes w_k$.

This condition implies that $\check{C}^{\bullet}(\mathcal{V})$ is isomorphic to

$$\check{C}^{\bullet}(\mathcal{V}) := \prod_{\alpha_0,\dots,\alpha_n} \mathbf{s}^n\Gamma(U_{\alpha_0}\cap\dots\cap U_{\alpha_n}, \mathcal{V})\,, \tag{B.8}$$

where the product is taken over all n-tuples $(\alpha_0, \dots, \alpha_n) \in \mathcal{I}$ for which $\alpha_i \neq \alpha_j$ if $i \neq j$.

The differential on (B.8) is the sum of the differential $\partial_{\mathcal{V}}$ coming from $\mathcal{V}$ and the Čech differential

$$\check{\partial}(f)_{\alpha_0 \alpha_1 \dots \alpha_n} = \sum_{j=0}^{n} (-1)^{|f|+j} f_{\alpha_0 \dots \widehat{\alpha_j} \dots \alpha_n} . \tag{B.9}$$

Remark B.1 Here we tacitly assume that the chosen cover of X is acyclic for the class of dg sheaves we consider. So we can always use the Cech resolution for computing the sheaf cohomology. In this article, X is usually a smooth algebraic variety considered with the Zariski topology. For our purposes, any cover of X by open affine subsets, each of which admits a global system of parameters, suffices.

Let $\mathcal{V}_1, \mathcal{V}_2$ be a pair of dg sheaves. The structure of associative algebra on (B.2) gives us the map of cochain complexes:

$$\mathrm{AW} : \check{C}^{\bullet}(\mathcal{V}_1) \otimes \check{C}^{\bullet}(\mathcal{V}_2) \to \check{C}^{\bullet}(\mathcal{V}_1 \otimes \mathcal{V}_2) \tag{B.10}$$

which is given by the formula:

$$\mathrm{AW}(f^1, f^2)_{\alpha_0 \dots \alpha_m} := \sum_{0 \le k \le m} (-1)^{|f^2| k} f^1_{\alpha_0 \dots \alpha_k} \otimes f^2_{\alpha_k \dots \alpha_m} . \tag{B.11}$$

We call AW the *Alexander-Whitney map.*

It is not hard to see that AW equips $\check{C}^{\bullet}$ with a natural structure of a monoidal functor from the category of dg sheaves to the category of cochain complexes. Unfortunately, the transformation AW is compatible with the braiding only up to homotopy. So, in general, AW cannot be used to pull an algebraic structure from a dg sheaf $\mathcal{V}$ to its Čech complex $\check{C}^{\bullet}(\mathcal{V})$.

It is the Thom-Sullivan normalization $\mathcal{N}^{\mathrm{TS}}$ [6], [37, Section 1], [45, Appendix A], which repairs this defect. Namely, $\mathcal{N}^{\mathrm{TS}}$ is a functor from the category of cosimplicial cochain complexes to the category of cochain complexes defined by the formula

$$\mathcal{N}^{\mathrm{TS}}(\mathfrak{S}) := \Omega^{\bullet}_{\mathrm{poly}}(\mathbf{\Delta}_*) \otimes_{\mathbf{\Delta}} \mathfrak{S}_* , \tag{B.12}$$

where $\mathfrak{S}$ is a cosimplicial cochain complex. (For example, $\mathfrak{S} = \check{C}(\mathcal{V})$ for a dg sheaf $\mathcal{V}$ on X).

Composing $\mathcal{N}^{\mathrm{TS}}$ with $\check{C}$, we get a functor from the category of dg sheaves on X to the category of cochain complexes. This composition $\mathcal{N}^{\mathrm{TS}} \circ \check{C}$ satisfies the following remarkable properties:

Property B.2 *The natural transformation*

$$\mu^{\mathrm{TS}} : \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_1) \otimes \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_2) \to \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_1 \otimes \mathcal{V}_2) \tag{B.13}$$

coming from the multiplication of exterior forms on geometric simplices equips $\mathcal{N}^{\mathrm{TS}} \circ \check{C}$ with a structure a monoidal functor from the category of dg sheaves to the category cochain complexes. This functor respects the braidings "on the nose".

Property B.3 *The map* (B.4) *induces a natural transformation*

$$\mathfrak{I}^{\check{C}} : \mathcal{N}^{\mathrm{TS}} \circ \check{C} \to \check{C}^{\bullet} \tag{B.14}$$

such that for every dg sheaf $\mathcal{V}$ *the map*

$$\mathfrak{I}^{\check{C}}_{\mathcal{V}} : \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}) \to \check{C}^{\bullet}(\mathcal{V})$$

induces an isomorphism on the level of cohomology.

Property B.4 *The functor* $\mathcal{N}^{\mathrm{TS}} \circ \check{C}$ *preserves quasi-isomorphisms.*

Property B.2 follows immediately from the construction and Property B.3 is a consequence of [6, Theorem 2.2]. Finally, Property (B.4) can be easily proved by using the descending filtration on $\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V})$ by the degree of exterior forms on geometric simplices.

In addition, we remark that the existence of maps (B.5) satisfying (B.6) implies that

Proposition B.5 *For every pair of dg sheaves* $\mathcal{V}_1$, $\mathcal{V}_2$ *on* X *the diagram*

$$\begin{tikzcd}
\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_1) \otimes \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_2) \arrow[r, "\mu^{\mathrm{TS}}"] \arrow[d, "\mathfrak{I}_{\mathcal{V}_1} \otimes \mathfrak{I}_{\mathcal{V}_2}"] & \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{V}_1 \otimes \mathcal{V}_2) \arrow[d, "\mathfrak{I}_{\mathcal{V}_1 \otimes \mathcal{V}_2}"] \\
\check{C}^{\bullet}(\mathcal{V}_1) \otimes \check{C}^{\bullet}(\mathcal{V}_2) \arrow[r, "\mathrm{AW}"] & \check{C}^{\bullet}(\mathcal{V}_1 \otimes \mathcal{V}_2)
\end{tikzcd} \tag{B.15}$$

commutes up to homotopy. □

B.2 Derived global sections of a dg sheaf of O-algebras

Let $\mathcal{A}$ be a dg sheaf on a topological space X. For $\mathcal{A}$ we form a dg operad $\mathrm{End}_{\mathcal{A}}$. As a graded vector space,

$$\mathrm{End}_{\mathcal{A}}(n) := \bigoplus_m \mathrm{Hom}^{(m)}(\mathcal{A}^{\otimes n}, \mathcal{A})\,, \tag{B.16}$$

where $\mathrm{Hom}^{(m)}(\mathcal{A}^{\otimes n}, \mathcal{A})$ consists of $\mathbb{K}$-linear maps of sheaves of degree m, and the differential

$$\partial : \mathrm{Hom}^{(m)}(\mathcal{A}^{\otimes n}, \mathcal{A}) \to \mathrm{Hom}^{(m+1)}(\mathcal{A}^{\otimes n}, \mathcal{A})$$

comes naturally from the differential on $\mathcal{A}$.

Let O be a dg operad. We recall that an O-algebra structure on a dg sheaf $\mathcal{A}$ is a map of dg operads

$$O \to \mathrm{End}_{\mathcal{A}}\,. \tag{B.17}$$

The monoidal structure on the functor $\mathcal{N}^{\mathrm{TS}} \circ \check{C}$ gives us a canonical map of dg operads

$$\mathrm{End}_{\mathcal{A}} \to \mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}\,. \tag{B.18}$$

Hence, for every dg sheaf $\mathcal{A}$ of O-algebras the cochain complex

$$\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A}) \tag{B.19}$$

is naturally an algebra over O.

We call the O-algebra (B.19) the *algebra of derived global sections of* $\mathcal{A}$.

B.3 Deformation complex of a sheaf of O-algebras

Deformation complex of an O-algebra admits a generalization to the setting of sheaves. We briefly describe this generalization here and refer the reader for more details to [23].

Let $\mathcal{A}$ be a dg sheaf on a topological space X equipped with an algebra structure over O. According to Subsection B.2, the cochain complex (B.19) is naturally an algebra over the operad O and hence an algebra over the operad $\mathrm{Cobar}(C)$.

Using this $\mathrm{Cobar}(C)$-algebra structure on (B.19), we get a degree 1 coderivation

$$Q^{\mathrm{TS}} \in \mathsf{coDer}'\Big(C\big(\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})\big)\Big) \tag{B.20}$$

of the "cofree" C-coalgebra

$$C\big(\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})\big)$$

satisfying the Maurer-Cartan equation.

Using this coderivation Q^{TS}, we equip the graded Lie algebra

$$\mathsf{coDer}\Big(C\big(\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})\big)\Big) \tag{B.21}$$

with the differential $\partial + [Q^{\mathrm{TS}}, \ ]$.

Definition B.6 *For a dg sheaf of O-algebras $\mathcal{A}$ we call the cochain complex* (B.21) *with the differential $\partial + [Q^{\mathrm{TS}}, \ ]$ the* deformation complex *of $\mathcal{A}$. We denote this complex by $\mathbf{Def}_O(\mathcal{A})$ or simply $\mathbf{Def}(\mathcal{A})$ when the operad O is clear from the context.*

According to [23] the deformation complex $\mathbf{Def}_O(\mathcal{A})$ is a homotopy invariant of $\mathcal{A}$. More precisely,

Theorem B.7 *Let $\mathcal{A}$, $\mathcal{B}$ be dg sheaves of O-algebras. If there exists a sequence of quasi-isomorphisms of sheaves of O-algebras connecting $\mathcal{A}$ to $\mathcal{B}$ then the dg Lie algebras $\mathbf{Def}_O(\mathcal{A})$ and $\mathbf{Def}_O(\mathcal{B})$ are quasi-isomorphic.*

Proof. This statement is a corollary of Theorem A.2 and Property B.4. Indeed, if two dg sheaves of O-algebras $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic then so are the dg O-algebras

$$\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})$$

and

$$\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{B}).$$

On the other hand, the deformation complex of the dg sheaf $\mathcal{A}$ (resp. $\mathcal{B}$) is the deformation complex of the dg O-algebra $\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})$ (resp. $\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{B})$).

Thus the desired statement follows from Theorem A.2. $\square$

B.4 A canonical homomorphism from $\mathsf{coDer}(C(\mathcal{A}))$ to $\mathbf{Def}_O(\mathcal{A})$

For a dg sheaf $\mathcal{A}$ we denote by $C(\mathcal{A})$ the dg sheaf of C-coalgebras "cofreely" cogenerated by $\mathcal{A}$. We also denote by

$$\mathsf{coDer}\big(C(\mathcal{A})\big) \tag{B.22}$$

the cochain complex of coderivations of $C(\mathcal{A})$. In other words, $\mathsf{coDer}\big(C(\mathcal{A})\big)$ consists of maps of sheaves

$$\mathcal{D} : C(\mathcal{A}) \to C(\mathcal{A}) \tag{B.23}$$

which are compatible with the C-coalgebra structure on $C(\mathcal{A})$ in the following sense:

$$\Delta_n \circ \mathcal{D} = \sum_{i=1}^{n} \big(\mathrm{id}_C \otimes \mathrm{id}_{\mathcal{A}}^{\otimes(i-1)} \otimes \mathcal{D} \otimes \mathrm{id}_{\mathcal{A}}^{n-i}\big) \circ \Delta_n \tag{B.24}$$

where Δ_n is the comultiplication map

$$\Delta_n : C(\mathcal{A}) \to \Big(C(n) \otimes \big(C(\mathcal{A})\big)^{\otimes n}\Big)^{S_n}.$$

The graded vector space (B.22) carries the obvious Lie bracket and the natural differential ∂ which comes from those on C and $\mathcal{A}$.

We denote by

$$\mathsf{coDer}'\big(C(\mathcal{A})\big) \tag{B.25}$$

the dg Lie subalgebra of (B.22) which consists of coderivations $\mathcal{D}$ satisfying the additional technical condition

$$\mathcal{D}\Big|_{\mathcal{A}} = 0\,.$$

Let us denote by $p_{\mathcal{A}}$ the canonical projection

$$p_{\mathcal{A}} : C(\mathcal{A}) \to \mathcal{A}\,.$$

It is not hard to see that the assignment

$$\mathcal{D} \mapsto p_{\mathcal{A}} \circ \mathcal{D}$$

gives us isomorphisms of dg Lie algebras

$$\mathsf{coDer}\big(C(\mathcal{A})\big) \cong \mathrm{Conv}(C, \mathrm{End}_{\mathcal{A}})\,, \tag{B.26}$$

and

$$\mathsf{coDer}'\big(C(\mathcal{A})\big) \cong \mathrm{Conv}(C_\circ, \mathrm{End}_{\mathcal{A}})\,, \tag{B.27}$$

where all the dg Lie algebras are considered with the differentials coming solely from the ones on C and $\mathcal{A}$.

Using the map of dg operads (B.18), we get the canonical morphism of dg Lie algebras

$$\Psi^{\heartsuit} : \mathrm{Conv}(C, \mathrm{End}_{\mathcal{A}}) \to \mathrm{Conv}\big(C, \mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}\big)\,, \tag{B.28}$$

where, again, the differentials come solely from the ones on C, $\mathcal{A}$, and $\mathcal{N}^{\mathrm{TS}} \circ \check{C}$.

If $\mathcal{A}$ is, in addition, a sheaf of O-algebras, then $\mathcal{A}$ is also a sheaf of Cobar(C)-algebras and $\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})$ is a Cobar(C)-algebra. Hence, due to [21, Proposition 5.2], we get the Maurer-Cartan element

$$\mathcal{Q} \in \mathrm{Conv}(C_\circ, \mathrm{End}_{\mathcal{A}}) \tag{B.29}$$

and hence a new differential $\partial + [\mathcal{Q}, \ ]$ on (B.26).

Let us recall that the graded Lie algebras

$$\mathrm{Conv}(C, \mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}) \tag{B.30}$$

and

$$\mathsf{coDer}\Big(C\big(\mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}\big)\Big)$$

are isomorphic. Furthermore, it is not hard to see that, the Maurer-Cartan element Q^{TS} (B.20) is related to $\mathcal{Q}$ via the equation

$$Q^{\mathrm{TS}} = \Psi^{\heartsuit}(\mathcal{Q})\,. \tag{B.31}$$

Therefore, the same map $\Psi^{\heartsuit}$ (B.28) gives us a morphism of dg Lie algebras

$$\Psi^{\heartsuit} : \Big(\mathrm{Conv}(C, \mathrm{End}_{\mathcal{A}}), \partial + [\mathcal{Q}, \ ]\Big) \to \Big(\mathrm{Conv}\big(C, \mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}\big), \partial + [Q^{\mathrm{TS}}, \ ]\Big) \tag{B.32}$$

and, since the target is canonically isomorphic to the deformation complex $\mathbf{Def}_O(\mathcal{A})$, $\Psi^{\heartsuit}$ can be viewed as a map of dg Lie algebras

$$\Psi^{\heartsuit} : \mathsf{coDer}(C(\mathcal{A})) \to \mathbf{Def}_O(\mathcal{A})\,. \tag{B.33}$$

B.5 A cocycle in $\mathsf{coDer}(C(\mathcal{A}))$ induces a derivation of the O-algebra $H^\bullet(X, \mathcal{A})$

Let us now adapt the construction of Subsection A.1 to the setting of sheaves. Just as in Subsection A.1, we assume that the operad O carries the zero differential.

Let $\mathcal{A}$ be a dg sheaf of O-algebras and $\mathcal{D}$ be a cocycle in $\mathsf{coDer}(C(\mathcal{A}))$, where $\mathsf{coDer}(C(\mathcal{A}))$ is considered with the differential $\partial + [\mathcal{Q}, \ ]$. According to the previous subsection,

$$\Psi^{\heartsuit}(\mathcal{D}) \tag{B.34}$$

is a cocycle in the deformation complex

$$\mathbf{Def}_O(\mathcal{A}) \cong \Big(\mathrm{Conv}\big(C, \mathrm{End}_{\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})}\big), \partial + [Q^{\mathrm{TS}}, \ ]\Big)\,. \tag{B.35}$$

Therefore, by Proposition A.4, the map

$$\mathfrak{B}_{\Psi^{\heartsuit}(\mathcal{D})} : \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A}) \to \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A}) \tag{B.36}$$

induces a derivation on the O-algebra

$$H^\bullet\Big(\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})\Big)\,. \tag{B.37}$$

On the other hand, the cochain complex $\mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A})$ computes the sheaf cohomology $H^\bullet(X, \mathcal{A})$. Hence, the map (B.36) induces a derivation of the O-algebra $H^\bullet(X, \mathcal{A})$.

For our purposes, we need an explicit way of computing this derivation in terms of conventional Čech cochains. This is given by the following proposition.

Proposition B.8 *Let $\mathcal{A}$ be a dg sheaf of O-algebras, $\mathcal{D}$ be a cocycle in $\mathsf{coDer}(C(\mathcal{A}))$, and v be a cochain in the Čech complex $\check{C}^\bullet(\mathcal{A})$. The formula*

$$\check{\mathfrak{B}}_{\mathcal{D}}(v)_{\alpha_0\alpha_1\dots\alpha_m} := \mathcal{D}(v_{\alpha_0\alpha_1\dots\alpha_m}) \tag{B.38}$$

defines degree $|\mathcal{D}|$ map

$$\check{\mathfrak{B}}_{\mathcal{D}} : \check{C}^\bullet(\mathcal{A}) \to \check{C}^\bullet(\mathcal{A})$$

which intertwines the differentials and such that the corresponding map

$$H^\bullet(X, \mathcal{A}) \to H^\bullet(X, \mathcal{A})$$

coincides with the derivation induced by (B.36)*.*

Proof. The compatibility of $\check{\mathfrak{B}}_{\mathcal{D}}$ with the differentials follows easily from the fact that $\mathcal{D}$ is a cocycle in $\mathsf{coDer}(C(\mathcal{A}))$.

Next, unfolding the definitions, we see that the diagram

$$\begin{tikzcd} \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A}) \arrow[r, "\mathfrak{B}_{\Psi^{\heartsuit}(\mathcal{D})}"] \arrow[d, "\mathfrak{I}_{\mathcal{A}}"] & \mathcal{N}^{\mathrm{TS}} \circ \check{C}(\mathcal{A}) \arrow[d, "\mathfrak{I}_{\mathcal{A}}"] \\ \check{C}^\bullet(\mathcal{A}) \arrow[r, "\check{\mathfrak{B}}_{\mathcal{D}}"] & \check{C}^\bullet(\mathcal{A}) \end{tikzcd} \tag{B.39}$$

commutes. So the second claim of the proposition follows as well. □

C Operations of twisting

In this section, we recall twisting of $\Lambda\mathsf{Lie}$-algebras and Gerstenhaber algebras by Maurer-Cartan elements. We also extend the twisting operation to a subspace of cochains in the deformation complex of a $\Lambda\mathsf{Lie}$-algebra and a Gerstenhaber algebra. For more details about the twisting procedure we refer the reader to [24].

C.1 Twisting operation for Chevalley-Eilenberg cochains

Let $\mathcal{V}$ be a dg $\Lambda\mathsf{Lie}$-algebra equipped with a complete descending filtration

$$\mathcal{V} \supset \dots \supset \mathcal{F}_0\mathcal{V} \supset \mathcal{F}_1\mathcal{V} \supset \mathcal{F}_2\mathcal{V} \supset \dots, \tag{C.1}$$

$$\mathcal{V} = \lim_k \mathcal{V} \,/\, \mathcal{F}_k\mathcal{V}, \tag{C.2}$$

which is compatible with the dg $\Lambda\mathsf{Lie}$-structure.

We say that a degree 2 vector $\alpha \in \mathcal{V}$ is a *Maurer-Cartan element*[21] if

$$\alpha \in \mathcal{F}_1\mathcal{V} \tag{C.3}$$

[21] Condition (C.3) is sometimes omitted.

and

$$\partial \alpha + \frac{1}{2}\{\alpha, \alpha\} = 0\,. \tag{C.4}$$

For every Maurer-Cartan element α of a $\Lambda\mathsf{Lie}$-algebra $\mathcal{V}$, the equations

$$\partial^{\alpha}_{\mathcal{V}} := \partial_{\mathcal{V}} + \{\alpha, ~\} \tag{C.5}$$

and

$$\{~,~\}^{\alpha} = \{~,~\} \tag{C.6}$$

define a new dg $\Lambda\mathsf{Lie}$-structure on $\mathcal{V}$.

We denote this new $\Lambda\mathsf{Lie}$-algebra by $\mathcal{V}^{\alpha}$ and say that $\mathcal{V}^{\alpha}$ is obtained from $\mathcal{V}$ via *twisting* by the Maurer-Cartan element α.

Now, for a given Maurer-Cartan element α of $\mathcal{V}$, we consider the following element of the completion of $\Lambda^2\mathsf{coCom}(\mathcal{V})$

$$\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{s}^2\,(\mathbf{s}^{-2}\,\alpha)^n \tag{C.7}$$

and define the subspace of coderivations $\mathcal{D} \in \mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V})\big)$ satisfying the additional condition[22]

$$\mathcal{D}\,\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = 0\,. \tag{C.8}$$

This subspace is obviously closed with respect to the commutator. Furthermore, we have the following theorem:

Theorem C.1 *Let $\mathcal{V}$ be a filtered $\Lambda\mathsf{Lie}$-algebra, α be a Maurer-Cartan element of $\mathcal{V}$, $p_{\mathcal{V}} : \Lambda^2\mathsf{coCom}(\mathcal{V}) \to \mathcal{V}$ be the canonical projection, and $\mathcal{Q}$ (resp. $\mathcal{Q}^{\alpha}$) be the codifferential on $\Lambda^2\mathsf{coCom}(\mathcal{V})$ (resp. $\Lambda^2\mathsf{coCom}(\mathcal{V}^{\alpha})$) corresponding to the dg $\Lambda\mathsf{Lie}$-structures on $\mathcal{V}$ (resp. $\mathcal{V}^{\alpha}$). Let us also denote by*

$$\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V})\big)_{\alpha} \tag{C.9}$$

the subspace of coderivations of $\Lambda^2\mathsf{coCom}(\mathcal{V})$ satisfying condition (C.8)*. Then*

i) *Condition* (C.8) *on coderivations is equivalent to*

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, p_{\mathcal{V}} \circ \mathcal{D}\left(\mathbf{s}^2\,(\mathbf{s}^{-2}\,\alpha)^n\right) = 0\,. \tag{C.10}$$

ii) *The codifferential $\mathcal{Q}$ satisfies Condition* (C.8)*.*

iii) *For every coderivation $\mathcal{D}$ in* (C.9) *the operation*

$$e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha} : \Lambda^2\mathsf{coCom}(\mathcal{V}) \to \Lambda^2\mathsf{coCom}(\mathcal{V}) \tag{C.11}$$

is a coderivation of $\Lambda^2\mathsf{coCom}(\mathcal{V})$.

[22] We tacitly assume that our coderivations are compatible with the filtration on $\Lambda^2\mathsf{coCom}(\mathcal{V})$ coming from (C.1).

iv) *The codifferential $\mathscr{Q}^\alpha$ is related to $\mathscr{Q}$ by the formula:*

$$\mathscr{Q}^\alpha \ = \ e^{-\mathbf{s}^{-2}\,\alpha}\,\mathscr{Q}\,e^{\mathbf{s}^{-2}\,\alpha}\,. \tag{C.12}$$

v) *The subspace* (C.9) *is a subcomplex of the deformation complex for $\mathcal{V}$. Furthermore, the assignment*

$$\mathcal{D} \ \mapsto \ \mathcal{D}^\alpha = e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha} \tag{C.13}$$

defines a map of cochain complexes

$$\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V})\big)_\alpha \ \to \ \mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V}^\alpha)\big) \tag{C.14}$$

from (C.9) *to the deformation complex* $\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V}^\alpha)\big)$ *of the* $\Lambda\mathsf{Lie}$*-algebra* $\mathcal{V}^\alpha$.

Proof. Using the compatibility of $\mathcal{D}$ with the comultiplication on $\Lambda^2\mathsf{coCom}(\mathcal{V})$, it is not hard to show that

$$\mathcal{D}\,\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = e^{\mathbf{s}^{-2}\,\alpha}\,p_{\mathcal{V}} \circ \mathcal{D}\big(\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1)\big)\,. \tag{C.15}$$

Hence, (C.8) is equivalent to (C.10) and claim **i)** follows.

Since α satisfies the Maurer-Cartan equation (C.4), we have

$$p_{\mathcal{V}} \circ \mathscr{Q}\,\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = 0\,.$$

Thus, claim **i)** implies claim claim **ii)**.

To prove claim **iii)**, we recall that the comultiplication Δ on

$$\Lambda^2\mathsf{coCom}(\mathcal{V}) = \mathbf{s}^2\underline{S}(\mathbf{s}^{-2}\,\mathcal{V})$$

is given by the formula:

$$\Delta(\mathbf{s}^2\,w_1 w_2 \dots w_n) = \tag{C.16}$$

$$\sum_{p=1}^{n-1}\ \sum_{\tau\in \mathrm{Sh}_{p,n-p}} (-1)^{\varepsilon(\tau,w_1,\dots,w_n)} \big(\mathbf{s}^2\,w_{\tau(1)}\dots w_{\tau(p)}\big) \ \otimes \ \big(\mathbf{s}^2\,w_{\tau(p+1)}\dots w_{\tau(n)}\big)\,,$$

$$w_1, w_2, \dots, w_n \in \mathbf{s}^{-2}\,\mathcal{V}\,,$$

where the sign factor $(-1)^{\varepsilon(\tau,w_1,\dots,w_n)}$ is determined by the usual Koszul rule of signs.

Let us extend $\mathcal{D}$ to the space of the full symmetric algebra

$$\mathbf{s}^2 S(\mathbf{s}^{-2}\,\mathcal{V}) \tag{C.17}$$

by requiring that

$$\hat{\mathcal{D}}(\mathbf{s}^2\,1) = 0\,. \tag{C.18}$$

Then $\mathcal{D}$ respects the following comultiplication on (C.17)

$$\widetilde{\Delta}\,(\mathbf{s}^2\,1) = \mathbf{s}^2\,1 \ \otimes \ \mathbf{s}^2\,1\,,$$

$$\widetilde{\Delta}\,(\mathbf{s}^2\,w_1\dots w_n) = \tag{C.19}$$

$$\mathbf{s}^2\, 1 \otimes (\mathbf{s}^2\, w_1 \dots w_n) + \sum_{p=1}^{n-1} \sum_{\tau \in \mathrm{Sh}_{p,n-p}} (-1)^{\varepsilon(\tau, w_1, \dots, w_n)} (\mathbf{s}^2\, w_{\tau(1)} \dots w_{\tau(p)}) \ \otimes \ (\mathbf{s}^2\, w_{\tau(p+1)} \dots w_{\tau(n)})$$

$$+ (\mathbf{s}^2\, w_1 \dots w_n) \otimes \mathbf{s}^2\, 1\,, \qquad w_1, \dots, w_n \in \mathbf{s}^{-2}\, \mathcal{V}\,, \qquad n \geq 1$$

in the sense of the identity

$$\widetilde{\Delta} \circ \mathcal{D} = (\mathcal{D} \otimes \mathrm{id} \ + \ \mathrm{id} \otimes \mathcal{D}) \circ \widetilde{\Delta}\,. \tag{C.20}$$

A direct computation shows that

$$\widetilde{\Delta}\left(\mathbf{s}^2\, e^{\mathbf{s}^{-2}\,\alpha}\right) = \mathbf{s}^2\, e^{\mathbf{s}^{-2}\,\alpha} \ \otimes \ \mathbf{s}^2\, e^{\mathbf{s}^{-2}\,\alpha}\,, \tag{C.21}$$

and the operation

$$W \ \mapsto \ e^{\mathbf{s}^{-2}\,\alpha}\, W \ : \ \mathbf{s}^2 S(\mathbf{s}^{-2}\, \mathcal{V}) \to \mathbf{s}^2 \hat{S}(\mathbf{s}^{-2}\, \mathcal{V}) \tag{C.22}$$

satisfies the identity

$$\widetilde{\Delta}\left(e^{\mathbf{s}^{-2}\,\alpha}\, W\right) = (e^{\mathbf{s}^{-2}\,\alpha} \otimes e^{\mathbf{s}^{-2}\,\alpha})\ \widetilde{\Delta}\,(W)\,. \tag{C.23}$$

Hence the operation

$$e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha} \ : \ \mathbf{s}^2 S(\mathbf{s}^{-2}\, \mathcal{V}) \to \mathbf{s}^2 \hat{S}(\mathbf{s}^{-2}\, \mathcal{V}) \tag{C.24}$$

satisfy the identity

$$\widetilde{\Delta} \circ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \ = \ \Big(\left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \otimes \mathrm{id} \ + \ \mathrm{id} \otimes \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right)\Big) \circ \widetilde{\Delta}\,. \tag{C.25}$$

On the other hand, Δ is related to $\widetilde{\Delta}$ by the formula

$$\Delta(W) = \widetilde{D}\,(W) - \mathbf{s}^2\, 1 \otimes W - W \otimes \mathbf{s}^2\, 1\,, \qquad \forall \ \ W \in \mathbf{s}^2 \underline{S}(\mathbf{s}^{-2}\, \mathcal{V})\,. \tag{C.26}$$

Therefore, using (C.18), (C.25), and (C.26), we get

$$\Delta \circ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W = \widetilde{\Delta} \circ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W$$

$$-\mathbf{s}^2 1 \otimes \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W - \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W \otimes \mathbf{s}^2 1 =$$

$$\Big(\left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \ \otimes \ \mathrm{id} \ \ + \ \ \mathrm{id} \ \otimes \ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right)\Big) \circ \widetilde{\Delta}\,(W)$$

$$-\mathbf{s}^2 1 \otimes \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W - \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W \otimes \mathbf{s}^2 1 =$$

$$\Big(\left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \ \otimes \ \mathrm{id} \ \ + \ \ \mathrm{id} \ \otimes \ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right)\Big) \circ \Delta(W)$$

$$+\Big(\left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \ \otimes \ \mathrm{id} \ \ + \ \ \mathrm{id} \ \otimes \ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right)\Big)(W \otimes \mathbf{s}^2\, 1 + \mathbf{s}^2\, 1 \otimes W)$$

$$-\mathbf{s}^2 1 \otimes \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W - \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) W \otimes \mathbf{s}^2 1 =$$

$$\Big(\left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right) \ \otimes \ \mathrm{id} \ \ + \ \ \mathrm{id} \ \otimes \ \left(e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\, e^{\mathbf{s}^{-2}\,\alpha}\right)\Big) \circ \Delta(W)$$

$$+ W \ \otimes \ e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\left(\mathbf{s}^2 (e^{\mathbf{s}^{-2}\,\alpha} - 1)\right) \ \ + \ \ e^{-\mathbf{s}^{-2}\,\alpha}\, \mathcal{D}\left(\mathbf{s}^2 (e^{\mathbf{s}^{-2}\,\alpha} - 1)\right) \ \otimes \ W\,.$$

Thus condition (C.8) implies that the operation (C.11) is indeed a derivation of $\Lambda^2\mathsf{coCom}(\mathcal{V})$.

Let us now prove claim **iv)**. Due to claim **iii)** the operation

$$e^{-\mathbf{s}^{-2}\,\alpha}\,\mathscr{Q}\,e^{\mathbf{s}^{-2}\,\alpha}$$

is a coderivation of $\Lambda^2\mathsf{coCom}(\mathcal{V})$. Hence, it suffices to show that

$$p_{\mathcal{V}} \circ \big(\mathscr{Q}\big)\, e^{\mathbf{s}^{-2}\,\alpha} = p_{\mathcal{V}} \circ \mathscr{Q}^{\alpha}\,. \tag{C.27}$$

Equation (C.27) directly follows from (C.5) and (C.6). Thus claim **iv)** follows.

Claim **v)** is now a straightforward consequence of claims **ii)** - **iv)**.

Theorem C.1 is proved. □

We say that the cochain $\mathcal{D}^{\alpha}$ in (C.13) is obtained from $\mathcal{D}$ via *twisting* by the Maurer-Cartan element α.

Theorem C.1 has the following corollary

Corollary C.2 *Let $\mathcal{V}$ be a filtered $\Lambda\mathsf{Lie}$-algebra, α be a Maurer-Cartan element of $\mathcal{V}$, and $\mathcal{V}^{\alpha}$ be the $\Lambda\mathsf{Lie}$-algebra which is obtained from $\mathcal{V}$ via twisting by α. If $\mathcal{D}$ is a cochain in the deformation complex for $\mathcal{V}$ satisfying the condition*

$$\mathcal{D}\big(\mathbf{s}^2(\mathbf{s}^{-2}\,\alpha)^n\big) = 0 \qquad \forall \;\; n \ge 1 \tag{C.28}$$

then

- *the operator*
$$\mathcal{D}^{\alpha} := e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha} : \Lambda^2\mathsf{coCom}(\mathcal{V}) \to \Lambda^2\mathsf{coCom}(\mathcal{V}) \tag{C.29}$$
is a cochain in the deformation complex
$$\Big(\mathsf{coDer}\big(\Lambda^2\mathsf{coCom}(\mathcal{V}^{\alpha})\big),\;\mathscr{Q}^{\alpha}\Big) \tag{C.30}$$
for $\mathcal{V}^{\alpha}$;

- *we have*
$$[\mathscr{Q},\mathcal{D}]^{\alpha} = [\mathscr{Q}^{\alpha},\mathcal{D}^{\alpha}]\,; \tag{C.31}$$

- *finally, for every $\Lambda\mathsf{Lie}_{\infty}$-derivation*[23] *$\mathcal{D}$ of $\mathcal{V}$ satisfying* (C.28)*, the cochain $\mathcal{D}^{\alpha}$* (C.29) *is a $\Lambda\mathsf{Lie}_{\infty}$-derivation of $\mathcal{V}^{\alpha}$.*

□

[23]Recall that degree zero cocycles in the deformation complex of a $\Lambda\mathsf{Lie}$-algebra $\mathcal{V}$ are called $\Lambda\mathsf{Lie}_{\infty}$-derivations of $\mathcal{V}$.

C.2 Twisting operation for cochains in the deformation complex of a Gerstenhaber algebra

We now assume that $\mathcal{V}$ is a dg Gerstenhaber algebra equipped with a complete descending filtration

$$\mathcal{V} \supset \dots \supset \mathcal{F}_0\mathcal{V} \supset \mathcal{F}_1\mathcal{V} \supset \mathcal{F}_2\mathcal{V} \supset \dots, \tag{C.32}$$

$$\mathcal{V} = \lim_k \mathcal{V} \,/\, \mathcal{F}_k\mathcal{V}, \tag{C.33}$$

which is compatible with the differential ∂, the $\Lambda\mathsf{Lie}$-bracket $\{\ ,\ \}$ and the multiplication $\cdot$ on $\mathcal{V}$.

Since every Gerstenhaber algebra is also a $\Lambda\mathsf{Lie}$-algebra, we have the notion of Maurer-Cartan elements in $\mathcal{V}$. Furthermore, given a Maurer-Cartan element α of $\mathcal{V}$, we denote by $\mathcal{V}^\alpha$ the dg Gerstenhaber algebra which is obtained from $\mathcal{V}$ via twisting by α. In other words, $\mathcal{V}^\alpha = \mathcal{V}$ as the graded vector space, the differential ∂^α on $\mathcal{V}^\alpha$ is given by equation (C.5). Finally $\mathcal{V}^\alpha$ and $\mathcal{V}$ share the same $\Lambda\mathsf{Lie}$-bracket $\{\ ,\ \}$ and the same multiplication $\cdot$.

Just as for $\Lambda\mathsf{Lie}$-algebras, we consider the following element of the completion of $\mathsf{Ger}^\vee(\mathcal{V})$

$$\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = \sum_{n=1}^{\infty} \frac{1}{n!}\mathbf{s}^2\,(\mathbf{s}^{-2}\,\alpha)^n \tag{C.34}$$

and define the subspace of coderivations $\mathcal{D} \in \mathsf{coDer}\big(\mathsf{Ger}^\vee(\mathcal{V})\big)$ satisfying the additional condition[24]

$$\mathcal{D}\,\mathbf{s}^2\,(e^{\mathbf{s}^{-2}\,\alpha} - 1) = 0\,. \tag{C.35}$$

This subspace is obviously closed with respect to the commutator.

We now present the following analogue of Theorem C.1

Theorem C.3 *Let $\mathcal{V}$ be a filtered Gerstenhaber algebra, α be a Maurer-Cartan element of $\mathcal{V}$, $p_{\mathcal{V}} : \mathsf{Ger}^\vee(\mathcal{V}) \to \mathcal{V}$ be the canonical projection, and $\mathcal{Q}$ (resp. $\mathcal{Q}^\alpha$) be the codifferential on $\mathsf{Ger}^\vee(\mathcal{V})$ (resp. $\mathsf{Ger}^\vee(\mathcal{V}^\alpha)$) corresponding to the dg Ger-structures on $\mathcal{V}$ (resp. $\mathcal{V}^\alpha$). Let us also denote by*

$$\mathsf{coDer}\big(\mathsf{Ger}^\vee(\mathcal{V})\big)_\alpha \tag{C.36}$$

the subspace of coderivations of $\mathsf{Ger}^\vee(\mathcal{V})$ satisfying condition (C.35). *Then*

i) *Condition* (C.35) *on coderivations is equivalent to*

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, p_{\mathcal{V}} \circ \mathcal{D}\,\big(\mathbf{s}^2\,(\mathbf{s}^{-2}\,\alpha)^n\big) = 0\,. \tag{C.37}$$

ii) *The codifferential $\mathcal{Q}$ satisfies Condition* (C.35).

iii) *For every coderivation $\mathcal{D}$ in* (C.36) *the operation*

$$e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha} : \mathsf{Ger}^\vee(\mathcal{V}) \to \mathsf{Ger}^\vee(\mathcal{V}) \tag{C.38}$$

is a coderivation of $\mathsf{Ger}^\vee(\mathcal{V})$.

[24]We tacitly assume that our coderivations are compatible with the filtration on $\mathsf{Ger}^\vee(\mathcal{V})$ coming from (C.32).

iv) *The codifferential $\mathscr{Q}^\alpha$ is related to $\mathscr{Q}$ by the formula:*

$$\mathscr{Q}^\alpha \;=\; e^{-\mathbf{s}^{-2}\,\alpha}\,\mathscr{Q}\,e^{\mathbf{s}^{-2}\,\alpha}\,. \tag{C.39}$$

v) *The subspace* (C.36) *is a subcomplex of the deformation complex for the Gerstenhaber algebra $\mathcal{V}$. Furthermore, the assignment*

$$\mathcal{D} \;\mapsto\; \mathcal{D}^\alpha = e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha} \tag{C.40}$$

defines a map of cochain complexes

$$\mathsf{coDer}\big(\mathsf{Ger}^\vee(\mathcal{V})\big)_\alpha \;\to\; \mathsf{coDer}\big(\mathsf{Ger}^\vee(\mathcal{V}^\alpha)\big) \tag{C.41}$$

from (C.36) *to the deformation complex* $\mathsf{coDer}\big(\mathsf{Ger}^\vee(\mathcal{V}^\alpha)\big)$ *of the Gerstenhaber algebra $\mathcal{V}^\alpha$.*

Proof. Proofs of all these statements are obtained by incorporating only minor modifications in the corresponding proof of Theorem C.1.

The only exception is probably claim **iii)**. In this case, we also have to prove that

$$\mathcal{D}^\alpha = e^{-\mathbf{s}^{-2}\,\alpha}\,\mathcal{D}\,e^{\mathbf{s}^{-2}\,\alpha}$$

respects the cobracket

$$\Delta_{\{\,,\,\}} : \mathsf{Ger}^\vee(\mathcal{V}) \to \mathsf{Ger}^\vee(\mathcal{V}) \otimes \mathsf{Ger}^\vee(\mathcal{V})$$

on $\mathsf{Ger}^\vee(\mathcal{V})$ in the sense of the equation

$$\Delta_{\{\,,\,\}} \circ \mathcal{D}^\alpha = (-1)^{|\mathcal{D}|}(\mathcal{D}^\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{D}^\alpha) \circ \Delta_{\{\,,\,\}}\,. \tag{C.42}$$

To prove this fact we observe that for any degree 2 vector $\alpha \in \mathcal{V}$ the operation

$$W \mapsto \mathbf{s}^{-2}\alpha\, W : \mathsf{Ger}^\vee(\mathcal{V}) \to \mathsf{Ger}^\vee(\mathcal{V})$$

is a degree 0 coderivation with respect to the cobracket $\Delta_{\{\,,\,\}}$. Hence for every vector W in the completion $\mathsf{Ger}^\vee(\mathcal{V})^\wedge$ of $\mathsf{Ger}^\vee(\mathcal{V})$ we have

$$\Delta_{\{\,,\,\}}\big(e^{\mathbf{s}^{-2}\alpha}\,W\big) = e^{\mathbf{s}^{-2}\alpha} \otimes e^{\mathbf{s}^{-2}\alpha}\,\big(\Delta_{\{\,,\,\}}W\big)\,. \tag{C.43}$$

Thus (C.42) indeed holds.

The compatibility of $\mathcal{D}^\alpha$ with the comultiplication on $\mathsf{Ger}^\vee(\mathcal{V})$ is proved in the same way as for the case of $\Lambda\mathsf{Lie}$-algebras. □

References

[1] A. Alekseev and C. Torossian, The Kashiwara-Vergne conjecture and Drinfeld's associators, Ann. of Math. (2) **175**, 2 (2012) 415–463; arXiv:0802.4300.

[2] Johan Alm and Sergei Merkulov, Grothendieck-Teichmueller group and Poisson cohomologies, arXiv:1203.5933.

[3] M. F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. **85** (1957) 181–207.

[4] S. Barannikov and M. Kontsevich, Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields, Internat. Math. Res. Notices, **4** (1998) 201–215; arXiv:alg-geom/9710032.

[5] C. Berger and B. Fresse, Combinatorial operad actions on cochains, Math. Proc. Cambridge Philos. Soc. **137**, 1 (2004) 135–174.

[6] A. K. Bousfield and V. K. A. M. Gugenheim, On *PL* de Rham theory and rational homotopy type, Mem. Amer. Math. Soc. **8**, 179 (1976) ix+94 pp.

[7] F. Brown, Mixed Tate motives over S$pec(\mathbb{Z})$, Annals of Math., **175**, 2 (2012) 949–976; arXiv:1102.1312.

[8] P. Brückmann, Zur Kohomologie von projektiven Hyperflachen. Beitrage zur Algebra und Geometrie 2, 87–101 (1974).

[9] P. Brückmann, Zur Kohomologie von vollständigen Durchschnitten mit Koeffizienten in der Garbe der Keime der Differentialformen, Mathematische Nachrichten, **71**, 1 (1976) 203–210.

[10] P. Brückmann, Zur Kohomologie von vollständigen Durchschnitten mit Koeffizienten in der Garbe der Keime der Differentialformen II, Mathematische Nachrichten, **77**, 1 (1977) 307–318.

[11] D. Calaque, A PBW theorem for inclusions of (sheaves of) Lie algebroids, arXiv:1205.3214.

[12] D. Calaque, A. Căldăraru and J. Tu, PBW for an inclusion of Lie algebras, arXiv:1010.0985.

[13] D. Calaque and M. Van den Bergh, Hochschild cohomology and Atiyah classes, Adv. Math. **224**, 5 (2010) 1839–1889; arXiv:0708.2725.

[14] A. Căldăraru, The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism, Adv. Math. **194**, 1 (2005) 34–66.

[15] Z. Chen, M. Stiénon, and P. Xu, From Atiyah Classes to Homotopy Leibniz Algebras, arXiv:1204.1075.

[16] P. Deligne and A.B. Goncharov, Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. Ecole Norm. Sup. (4) **38**, 1 (2005) 1–56.

[17] V. A. Dolgushev, A Formality theorem for Hochschild chains, Adv. Math. **200**, 1 (2006) 51–101.

[18] V. A. Dolgushev, Covariant and equivariant formality theorems, Adv. Math., **191**, 1 (2005) 147–177; math.QA/0307212.

[19] V. A. Dolgushev, Exhausting formal quantization procedures, accepted to *Proceedings of the XXX Workshop on Geometric Methods in Physics*; arXiv:1111.2797.

[20] V. A. Dolgushev, Stable formality quasi-isomorphisms for Hochschild cochains I, arXiv:1109.6031.

[21] V.A. Dolgushev and C.L. Rogers, Notes on algebraic operads, graph complexes, and Willwacher's construction, to appear in Contemp. Math. AMS; arXiv:1202.2937.

[22] V.A. Dolgushev, D.E. Tamarkin, and B.L. Tsygan, The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal, J. Noncomm. Geom. **1**, 1 (2007) 1-25; math.KT/0605141.

[23] V.A. Dolgushev and T.H. Willwacher, Deformation complex is a homotopy invariant of a homotopy algebra, In preparation.

[24] V.A. Dolgushev and T.H. Willwacher, Operadic Twisting – with an application to Deligne's conjecture, arXiv:1207.2180.

[25] V.G. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbb{Q}}/\mathbb{Q}$). (Russian) Algebra i Analiz **2**, 4 (1990) 149–181; translation in Leningrad Math. J. **2**, 4 (1991) 829–860.

[26] B. Fresse, Koszul duality of operads and homology of partition posets, in "Homotopy theory and its applications (Evanston, 2002)", Contemp. Math. **346** (2004) 115–215.

[27] H. Furusho, Four groups related to associators, arXiv:1108.3389.

[28] I.M. Gelfand and D.V. Fuchs, Cohomology of the algebra of formal vector fields, Izv. Akad. Nauk., Math. Ser. **34** (1970) 322–337 (In Russian).

[29] I.M. Gelfand and D.A. Kazhdan, Some problems of differential geometry and the calculation of cohomologies of Lie algebras of vector fields, Soviet Math. Dokl., **12**, 5 (1971) 1367–1370.

[30] E. Getzler and J.D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, hep-th/9403055.

[31] V. Ginzburg and M. Kapranov, Koszul duality for operads, Duke Math. J. **76**, 1 (1994) 203–272.

[32] A. Grothendieck, Sketch of a program, *London Math. Soc. Lect. Note Ser.,* **242**, Geometric Galois actions, 1, 5–48, Cambridge Univ. Press, Cambridge, 1997.

[33] V. Hinich, Tamarkin's proof of Kontsevich formality theorem, Forum Math. **15**, 4 (2003) 591–614; math.QA/0003052.

[34] V. Hinich and V. Schechtman, Homotopy limits of homotopy algebras, in: *Yu.I. Manin (ed), K-Theory, Arithmetic and Geometry,* Lectures Notes in Math. **1289**, Springer-Verlag, Berlin, 1987, 240–264.

[35] F. Hirzebruch, Topological Methods in Algebraic Geometry. 3rd edition, Springer-Verlag, Berlin-Heidelberg-New York, (1966)

[36] G. Hochschild, B. Kostant, and A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. **102** (1962) 383–408.

[37] R. Huebl and A. Yekutieli, Adelic Chern Forms and Applications, Amer. J. Math. **121**, 4 (1999) 797–839; arXiv:math/9811109v1.

[38] C. Jost, Globalizing L-infinity automorphisms of the Schouten algebra of polyvector fields, arXiv:1201.1392.

[39] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compositio Math. **115**, 1 (1999) 71–113; arXiv:alg-geom/9704009.

[40] M. Kontsevich, Formality conjecture, *Deformation theory and symplectic geometry* (Ascona, 1996), 139–156, Math. Phys. Stud., 20, Kluwer Acad. Publ., Dordrecht, 1997.

[41] M. Kontsevich, Operads and Motives in Deformation Quantization, Lett. Math. Phys. **48** (1999) 35–72; arXiv:math/9904055.

[42] P. A. B. Lecomte, P.W. Michor, and H. Schicketanz, The multigraded Nijenhuis-Richardson algebra, its universal property and applications, J. Pure Appl. Algebra **77**, 1 (1992) 87–102.

[43] N. Markarian, The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem. J. Lond. Math. Soc. (2) **79**, 1 (2009) 129–143.

[44] S. Merkulov and B. Vallette, Deformation theory of representations of prop(erad)s. I, II. J. Reine Angew. Math. **634** (2009) 51–106, J. Reine Angew. Math. **636** (2009) 123–174.

[45] M. Van den Bergh, On global deformation quantization in the algebraic case, J. Algebra **315**, 1 (2007) 326–395.

[46] T. Willwacher, Characteristic classes in deformation quantization, arXiv:1208.4249.

[47] T. Willwacher, M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra, arXiv:1009.1654.

[48] A. Yekutieli, Mixed resolutions, simplicial sections and unipotent group actions, Israel J. Math. **162** (2007) 1–27.

Department of Mathematics, Temple University,
Wachman Hall Rm. 638
1805 N. Broad St.,
Philadelphia PA, 19122 USA
E-mail address: **vald@temple.edu**

Mathematisches Institut
Georg-August Universität Göttingen
Bunsenstrasse 3-5
D-37073 Göttingen
Deutschland
E-mail address: **crogers@uni-math.gwdg.de**

Society of Fellows, Harvard University
78 Mount Auburn St.
Cambridge, MA 02138, USA
E-mail address: **t.willwacher@gmail.com**